Accelerated H^2-norm Approximation for Time-Delay Systems with Discrete Delays ⋆

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Abstract: For time-delay systems with a single delay, it was recently demonstrated that certain spectral discretizations satisfying a symmetry condition can give super-geometric convergence when approximating the H^2-norm [*SIAM J Numer Anal*, 62(6), 2529–48]. In this work, this effect is partially extended to time-delay systems with multiple discrete delays, by using splines instead of a single polynomial, seeing a significantly increased convergence rate and attaining super-geometric convergence for equidistant delays. By investigating the transfer function of both discretizations, intuitive grounds are provided for this accelerated convergence. Unlike a discretization using only one polynomial, all exponential functions corresponding to non-zero delays are approximated using a distinct rational function. Furthermore, these approximations match the exponential better in a qualitative sense, by having a constant modulus of one on the imaginary axis. Finally, for commensurate delays, a strong connection between the spline discretization and the polynomial discretization of a descriptor-system reduction to a single delay is shown, which informs a method with super-geometric convergence for these systems.

Keywords: H^2-norm approximation, robustness analysis, spectral methods, numerical analysis

1. INTRODUCTION

The H^2-norm is both an important metric of system resilience in robust control and commonly used in reduced order modelling. For a linear, time invariant, time-delay system, a practical approximation method for this norm was proposed by Vanbiervliet et al. (2011), where the H^2-norm of the original system is approximated by the exact H^2-norm of a delay-free approximation of this system.

They report third order algebraic convergence in the discretization degree when using the pseudospectral collocation method of Breda et al. (2005) with Chebyshev extrema nodes. In Provoost and Michiels (2024) the authors demonstrate an improvement to super-geometric convergence for systems with a single delay when using a Lanczos tau discretization subject to a symmetry condition instead, such as the original Legendre–tau discretization of Ito and Teglas (1986).

In this work, we show how this improvement can be partially extended to multiple discrete delays, by using a discretization based on a spline instead of a single polynomial—similar to what is done in Ito and Teglas (1987) and Breda et al. (2005)—yielding an efficient method for this class of systems.

For an exponentially stable system with transfer function G, the H^2-norm is given by

$$\|G\|_{H^2} = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \|G(i\omega)\|_F^2 \,\mathrm{d}\omega\right)^{\frac{1}{2}},$$

with $\|A\|_F = \sqrt{\operatorname{tr}(AA^*)}$ the Frobenius norm and i the imaginary unit. We shall approximate this norm for the following time-delay system

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \sum_{k=0}^{m} A_k \mathbf{x}(t-\tau_k) + B\mathbf{u}(t),\\ \mathbf{y}(t) &= C\mathbf{x}(t),\end{aligned} \tag{1}$$

with $\tau_0 = 0 < \tau_1 < \cdots < \tau_m < \infty$ discrete delays, $\mathbf{x}(t) \in \mathbb{C}^n$ the state, $\mathbf{u}(t) \in \mathbb{C}^p$ the input, and $\mathbf{y}(t) \in \mathbb{C}^q$ the output at time t, and $A_0, \ldots, A_m$, B, and C complex matrices with corresponding dimensions. The transfer function of this system is given by

$$G(s) = C\left(sI_n - \sum_{k=0}^{m} A_k e^{-\tau_k s}\right)^{-1} B.$$

In the next section we review the approximation method yielding said improved convergence for a single delay, the section thereafter will extend this spectral discretization to splines, which will allow for significantly increased convergence rates for multiple delays and even super-geometric rates when the delays are equidistant.

In the same section we show that the transfer function of the spline discretization, unlike the polynomial one, contains distinct rational approximations for each exponential function corresponding with a delay. We provide further

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intuitions for the observed improvement in convergence rate, by showing these approximants to have modulus one on the imaginary axis (subsection 3.1) and demonstrate a connection to the descriptor-system reformulation of a system with commensurate delays (subsection 3.2), allowing super-geometric convergence for this type of system.

2. DISCRETIZATION USING POLYNOMIALS

For systems without delay, i.e. $m = 0$, it is well known (see e.g. Zhou et al., 1995, Lemma 4.6) that

$$\|G\|_{H^2} = \sqrt{\mathrm{tr}(CVC^T)}, \tag{2}$$

with V the solution of the Lyapunov equation

$$A_0 V + V A_0^T = -BB^T.$$

Vanbiervliet et al. (2011) propose to use a spectral discretization to approximate the delay differential equation by an ordinary one and use the H^2-norm of the latter as an approximation of that of the former.

One such group of discretizations are the Lanczos tau methods, a straightforward generalization of the Legendre–tau method introduced by Ito and Teglas (1986) to arbitrary bases. By noting that the full state at time t—that is, the minimal information needed to uniquely define the forward solution of (1) from time t onward—is given by the history function

$$\xi_t : [-\tau_m, 0] \to \mathbb{C}^n, \quad \theta \mapsto \mathbf{x}(t+\theta),$$

we can rewrite the system (1) as an abstract Cauchy problem. To handle input in this reformulation, we explicitly decouple the current state from the history in a 'head-tail' representation, in following with Curtain and Zwart (1995).

Consider as state space

$$Z := \mathbb{C}^n \times L^2([-\tau_m, 0]; \mathbb{C}^n),$$

and let $\mathcal{A} : D(\mathcal{A}) \to Z$ be the differential operator with action

$$\mathcal{A}(\mathbf{z}, \zeta) = \left(\textstyle\sum_{k=0}^m A_k \zeta(-\tau_k), \tfrac{\mathrm{d}}{\mathrm{d}\theta}\zeta\right)$$

and domain

$$D(\mathcal{A}) = \big\{(\mathbf{z}, \zeta) \in Z : \zeta \in AC([-\tau_m, 0]; \mathbb{C}^n), \\ \tfrac{\mathrm{d}}{\mathrm{d}\theta}\zeta \in L^2([-\tau_m, 0]; \mathbb{C}^n), \mathbf{z} = \zeta(0)\big\}.$$

Next, define the operators $\mathcal{B} : \mathbb{C}^p \to Z$ and $\mathcal{C} : Z \to \mathbb{C}^q$ with actions

$$\mathcal{B}\mathbf{u} = (B\mathbf{u}, \mathbf{0}) \quad \text{and} \quad \mathcal{C}z = C\mathbf{z},$$

where $\mathbf{u} \in \mathbb{C}^p$ and $z = (\mathbf{z}, \zeta) \in Z$.

With these three operators we then obtain the infinite dimensional system

$$\dot{z}(t) = \mathcal{A}z(t) + \mathcal{B}\mathbf{u}(t),$$
$$\mathbf{y}(t) = \mathcal{C}z(t),$$

where $z(t) = (\mathbf{z}(t), \zeta_t) \in D(\mathcal{A})$. Solutions of this Cauchy problem and those of the original system are straightforwardly related by

$$\mathbf{z}(t) = \mathbf{x}(t) \quad \text{and} \quad \zeta_t(\theta) = \mathbf{x}(t+\theta) \quad \forall \theta \in [-\tau_m, 0].$$

In other words, ζ_t is pointwise equal to the history function ξ_t.

We can now discretize this reformulation, and hence also system (1), by approximating ζ_t by $\xi_{tN} : [-\tau_m, 0] \to \mathbb{C}^n$, a polynomial of degree N. On the right hand side, the

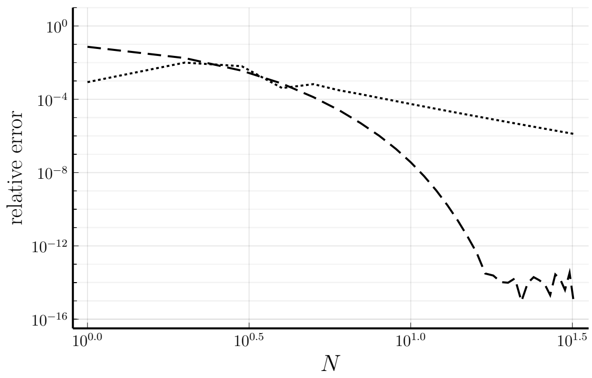

Fig. 1. Convergence of the H^2-norm of the system (1) with $m = 1$, $\tau_1 = 1$, $A_0 = \left(\begin{smallmatrix} -5 & 1 \\ 3 & -8 \end{smallmatrix}\right)$, $A_1 = \left(\begin{smallmatrix} -2 & 0 \\ -2 & 1 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$, and $C = \left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right)$, for Lanczos tau methods with different ϕ_N. The dotted line uses the Nth Jacobi $\left(-\frac{1}{2}, -\frac{3}{4}\right)$ polynomial,[1] which is non-symmetric, and the dashed line uses the Nth Chebyshev polynomial of the second kind, which is symmetric.

derivative, embedded in the action of $\mathcal{A}$, reduces the degree of the polynomial by one, correspondingly, we will need a similar reduction on the left hand side. The idea of a Lanczos tau method is to drop the $(N+1)$th, i.e. the final, coefficient of an expansion of ξ_{tN} in a degree-graded, orthogonal polynomial basis $\{\phi_j : [-\tau_m, 0] \to \mathbb{C}\}_{j=0}^N$ (Lanczos, 1938). That is, a basis with ϕ_j of degree j and $\langle \phi_k, \phi_j \rangle = 0$ iff $j \neq k$, for a suitable inner product $\langle \cdot, \cdot \rangle$. This yields, up to the choice of ϕ_N, the following family of delay-free approximations of (1):

$$\begin{pmatrix} \varepsilon_0 \\ \mathcal{T}_{\phi_N} \end{pmatrix} \dot{\xi}_{tN} = \begin{pmatrix} \sum_{k=0}^m A_k \varepsilon_{-\tau_k} \\ \mathcal{D} \end{pmatrix} \xi_{tN} + \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix} \mathbf{u}(t), \tag{3}$$
$$\mathbf{y}_N(t) = C\varepsilon_0 \xi_{tN},$$

with evaluation functionals $\varepsilon_\theta \xi = \xi(\theta)$, differentiation operator $\mathcal{D}\xi(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}\xi(\theta)$, and orthogonal projection $\mathcal{T}_{\phi_N}$ such that

$$(\mathcal{T}_{\phi_N}\xi)_j = (\xi)_j - \langle (\xi)_j, \phi_N \rangle \frac{\phi_N}{\|\phi_N\|^2}, \quad j = 1, \ldots, n.$$

By expressing ξ_{tN} in a polynomial basis, this discretization can be written in a matrix-vector format. How to obtain coordinate expressions of the operators is detailed further in Boyd (2001, Chapters 3 and 21).

To use this discretization to approximate the H^2-norm of the system (1), we must show that the implicit formulation of (3) can be recast as a standard delay-free system. To this end, note that $\ker(\mathcal{T}_{\phi_N}) = \mathrm{span}(I_n \phi_N)$. Furthermore, a basic result on orthogonal polynomials states that their zeroes lie in the interior of their domain (Szegő, 1939, Theorem 3.3.1), hence $\phi_N(0) \neq 0$. As a consequence,

$$\ker(\varepsilon_0) \cap \ker(\mathcal{T}_{\phi_N}) = \{\mathbf{0}\}$$

and $\left(\begin{smallmatrix} \varepsilon_0 \\ \mathcal{T}_{\phi_N} \end{smallmatrix}\right)$ is invertible.

[1] The Jacobi polynomial $P_k^{(\alpha,\beta)}$ on the interval $[-1, 1]$ is the unique polynomial of degree k which is orthogonal to all polynomials of lower degree with respect to the inner product $\langle \phi, \psi \rangle = \int_{-1}^1 (1-\theta)^\alpha (1+\theta)^\beta \, \phi(\theta)\psi(\theta) \, \mathrm{d}\theta$ and satisfies $P_k^{(\alpha,\beta)}(1) = \binom{k+\alpha}{k}$. We can recover the kth Chebyshev polynomial of the second kind as $U_k = (k+1)\binom{k+\frac{1}{2}}{k}^{-1} P_k^{(\frac{1}{2},\frac{1}{2})}$.

We can thus use (2) to approximate

$$\|G\|_{H^2} \approx \|G_N\|_{H^2},$$

where G_N is the transfer function of the discretization (3).

As observed in Provoost and Michiels (2024) and illustrated on Figure 1, we see super-geometric convergence when we limit (1) to be a system with a single non-zero delay, i.e. $m = 1$, and choose a ϕ_N that is symmetric on the interval $[-\tau_m, 0]$, that is

$$\phi_N(-\tau_m - \theta) = (-1)^N \phi_N(\theta), \quad \forall \theta \in [-\tau_m, 0].$$

If $m > 1$, however, we fall back to the third order algebraic convergence reported in Vanbiervliet et al. (2011), as seen on the dashed curve of Figure 2. The aim of the next section is to extend this acceleration to arbitrary m. To do so, we first look at the transfer function of the discretization of this section.

From a straightforward extension of Proposition 3.1 of Provoost and Michiels (2024), we know that

$$G_N(s) = C\left(sI_n - \sum_{k=0}^{m} A_k r_N(s, -\tau_k)\right)^{-1} B,$$

where $r_N(s, \theta)$ is a rational function of s and polynomial in $\theta \in [-\tau_m, 0]$, such that

$$\begin{cases} r_N(s, 0) = 1, \\ \mathcal{D} r_N(s, \cdot\,) = s \mathcal{T}_{\phi_N} r_N(s, \cdot\,). \end{cases}$$

Whilst we are still unable to prove the super-geometric convergence seen in the single delay case, the fact that it only occurs for a symmetric basis implies that effects at the ends of the approximation interval have an important influence. In the case of multiple delays, however, $\theta \mapsto r_N(s, \theta)$ is evaluated in interior points. This thus hints at using a discretization where every delay interval $[\tau_k, \tau_{k-1}]$, $k = 1, \ldots, m$, has its own $r_N^{(k)}(s, \theta)$.

3. ACCELERATION USING SPLINES

As we will see, one way to achieve distinct rational approximants for each delay interval is by using a spline. In Ito and Teglas (1987) the authors already propose to use splines in a tau method for a system with discrete delays, where the spline is allowed to be discontinuous to prevent jump discontinuities from resulting in a poor approximation. In our application, however, the latter does not occur; similarly to the spline pseudospectral collocation method of Breda et al. (2005), we will thus require the spline to be continuous. Let Ξ_{tN} be a set of polynomials $\{\xi_{tN}^{(k)} : [-\tau_k, -\tau_{k-1}] \to \mathbb{C}^n\}_{k=1}^{m}$, each of degree N, with continuity conditions

$$\xi_{tN}^{(k)}(-\tau_k) = \xi_{tN}^{(k+1)}(-\tau_k), \quad k = 1, \ldots, m-1. \tag{4}$$

Analogous to the preceding section we can now discretize system (1), by expanding each $\xi_{tN}^{(k)}$ in an orthogonal, degree-graded basis $\{\phi_{k,j} : [-\tau_k, -\tau_{k-1}] \to \mathbb{C}\}_{j=0}^{N}$. Note, however, that the derivative of the continuous spline $\Xi_{tN}(\theta)$ with respect to θ is not, in general, continuous in θ, whilst the derivative with respect to t is. We must thus close the system by adding continuity conditions on the derivative with respect to θ, which results in equation (5) at the top of the next page. Through an analogous argument to last section, we can see that equation (5) is also, implicitly, a delay-free system.

Before computing the H^2-norm, we first take a closer look at what happens in the frequency domain. Taking the Laplace transform of discretization (5) gives:

$$\begin{cases} s\varepsilon_0^{(1)} \hat{\Xi}_{sN} = \left(A_0 \varepsilon_0^{(1)} + \displaystyle\sum_{k=1}^{m} A_k \varepsilon_{-\tau_k}^{(k)}\right) \hat{\Xi}_{sN} + B\hat{\mathbf{u}}(s), \\ s\mathcal{T}_{\phi_{k,N}}^{(k)} \hat{\Xi}_{sN} = \mathcal{D}^{(k)} \hat{\Xi}_{sN}, \qquad k = 1, \ldots, m, \\ s\left(\varepsilon_{-\tau_k}^{(k)} - \varepsilon_{-\tau_k}^{(k+1)}\right) \hat{\Xi}_{sN} = \mathbf{0}, \quad k = 1, \ldots, m-1, \\ \hat{\mathbf{y}}_N(s) = C\varepsilon_0^{(1)} \hat{\Xi}_{sN}, \end{cases} \tag{6}$$

where $\hat{f}(s)$ is the Laplace transform of $f(t)$.

Analogous to Proposition 3.1 of Provoost and Michiels (2024), we can prove that the transfer function of the spline discretization is given by

$$G_N^{\mathrm{spl}}(s) = C\left(sI_n - A_0 r_N^{(1)}(s, 0) - \sum_{k=1}^{m} A_k r_N^{(k)}(s, -\tau_k)\right)^{-1} B,$$

where $r_N^{(k)}(s, \theta)$ is a rational function of s and polynomial in $\theta \in [-\tau_k, -\tau_{k-1}]$, such that

$$\begin{cases} r_N^{(1)}(s, 0) = 1, \\ r_N^{(k)}(s, -\tau_{k-1}) = r_N^{(k-1)}(s, -\tau_{k-1}), \quad k = 2, \ldots, m, \\ \mathcal{D} r_N^{(k)}(s, \cdot\,) = s\mathcal{T}_{\phi_{k,N}} r_N^{(k)}(s, \cdot\,), \qquad k = 1, \ldots, m. \end{cases}$$

The bulk of the argument relies on noticing that we can write $\hat{\Xi}_{sN}$ in terms of the $r_N^{(k)}$. The conditions on the latter then end up corresponding to the middle two equations of (6).

As a corollary, this discretization has a distinct rational approximation $r_N^{(k)}(s, -\tau_k)$ of $s \mapsto e^{-\tau_k s}$ for each delay interval $[-\tau_k, -\tau_{k-1}]$, as we set out to achieve. Hence, the poles—and even the type, as we shall see in subsection 3.1—of the rational approximations need not be the same for all delay intervals, which is unlike the case with a single polynomial where only the zeroes can change.

To compute the H^2-norm, we will first need to treat a small technical artefact that occurs when using (5) directly to produce the Lyaponuv equation due to losing the domain condition (4). Note that the factor s in the third equation of (6) introduces a removable singularity at 0 in the transfer function, this corresponds to eigenvalues at 0 of the matrix pencil used in the Lyapunov equation. To solve the latter, all eigenvalues must have a negative real part; our method would hence fail. Luckily we can easily correct this issue by replacing this equation by the equivalent

$$s\left(\varepsilon_{-\tau_k}^{(k)} - \varepsilon_{-\tau_k}^{(k+1)}\right) \hat{\Xi}_{sN} = -\left(\varepsilon_{-\tau_k}^{(k)} - \varepsilon_{-\tau_k}^{(k+1)}\right) \hat{\Xi}_{sN},$$

which moves the removable singularity from 0 to -1 whilst still having an equivalent transfer function.

If we now approximate the H^2-norm of the original system using this modified discretization, we see an improvement to about fifth order algebraic convergence when using a symmetric $\phi_{k,N}$, as in Figure 2. If we take the delays equidistant, as in Figure 3, we even recover the

We can discretize system (1), using a continuous spline Ξ_{tN}, as

$$\begin{pmatrix} \varepsilon_0^{(1)} \\ \left(\mathcal{T}_{\phi_{k,N}}^{(k)}\right)_{k=1}^{m} \\ \left(\varepsilon_{-\tau_k}^{(k)} - \varepsilon_{-\tau_k}^{(k+1)}\right)_{k=1}^{m-1} \end{pmatrix} \dot{\Xi}_{tN} = \begin{pmatrix} A_0\varepsilon_0^{(1)} + \sum_{k=1}^{m} A_k \varepsilon_{-\tau_k}^{(k)} \\ \left(\mathcal{D}^{(k)}\right)_{k=1}^{m} \\ (\mathbf{0})_{k=1}^{m-1} \end{pmatrix} \Xi_{tN} + \begin{pmatrix} B \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \mathbf{u}(t), \tag{5}$$

$$\mathbf{y}_N(t) = C\varepsilon_0^{(1)}\Xi_{tN},$$

where $\mathcal{P}^{(k)}$ indicates that $\mathcal{P}$ is applied to segment $\xi_{tN}^{(k)}$ and $(\mathcal{P}_k)_{k=1}^{m}$ is the vertical concatenation of $\{\mathcal{P}_1, \ldots, \mathcal{P}_m\}$.

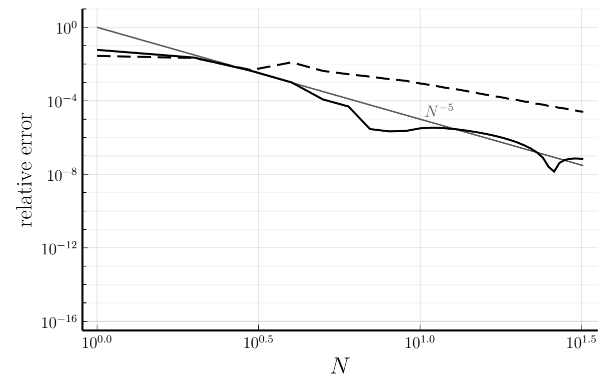

Fig. 2. Convergence of the H^2-norm of the system (1) with $m = 2$, $\tau_2 = 1.9$, $A_2 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and all other parameters as in Figure 1, for different Lanczos tau type methods with ϕ_N and $\phi_{k,N}$ the appropriately shifted Nth Chebyshev polynomial of the second kind, which is symmetric. The dashed line uses a single polynomial for the entire interval and the solid line uses a spline with a separate polynomial for each delay interval; the latter approximately follows the grey fifth order reference line.

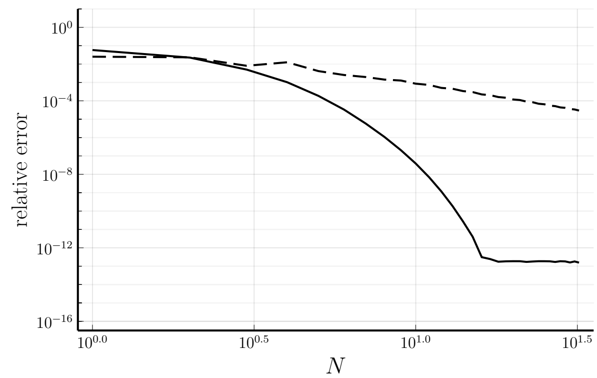

Fig. 3. The same experiment as in Figure 2 but with $\tau_2 = 2$. i.e. with equidistant delays.

super-geometric convergence of the single delay case. Furthermore, as systems with commensurate delays can be rewritten using a single delay, we can also recover super-geometric convergence for these systems. We return to this case in subsection 3.2.

As solving the Lyapunov equation dominates the time complexity, the method requires $\mathcal{O}\left(M^3\right)$ operations, with M the number of rows in the discretization (Hammarling, 1982). Using a spline gives a dependence on the number of delays m, and is hence more expensive than the single polynomial, requiring $\mathcal{O}\left(m^3 n^3 N^3\right)$ and $\mathcal{O}\left(n^3 N^3\right)$ operations, respectively. However, as this is only a constant factor for a given system, a spline approach will generally need less operations for the same level of accuracy, especially when high accuracy is required or when there are only a moderate number of delays.

Similarly to the polynomial Lanczos tau method, the spline version is also related to the spline extension of pseudospectral collocation as introduced in Breda et al. (2005). In their work, an ordinary differential equation is obtained directly by having both end points of a segment as collocation nodes, which allows the elimination of algebraic constraints. However, accelerated convergence is only obtained for a symmetric, orthogonal $\phi_{k,N}$. Whilst a Lanczos tau method corresponds to collocation of the relation $\frac{\mathrm{d}}{\mathrm{d}t}\zeta_t(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}\zeta_t(\theta)$ in the zeroes of $\phi_{k,N}$ (analogous to Provoost and Michiels, 2024, Theorem 4.1), orthogonal polynomials cannot have zeroes in the end points and hence we cannot collocate the derivative in the delays. Unfortunately, this simplification can thus not be used without reducing the convergence rate.

3.1 Magnitude on the imaginary axis

A final piece of intuition as to why the spline discretization outperforms the polynomial one, is the magnitude of the rational approximants of the exponential on the imaginary axis. Let $\rho_{k,N}(s,\theta)$ be a rational function of s and polynomial in $\theta \in [-\tau_k, -\tau_{k-1}]$, such that

$$\begin{cases} \rho_{k,N}(s, -\tau_{k-1}) = 1, \\ \mathcal{D}\rho_{k,N}(s, \cdot\,) = s\mathcal{T}_{\phi_{k,N}}\rho_{k,N}(s, \cdot\,). \end{cases}$$

It is then easy to see that the rational approximants of the exponential in G_N^{spl} are given by $r_N^{(1)} = \rho_{1,N}$ and

$$r_N^{(k)}(s,\theta) = r_N^{(k-1)}(s, -\tau_{k-1})\rho_{k,N}(s,\theta), \quad k = 2, \ldots, m.$$

As $\rho_{k,N}(s,\theta)$ is a type (N, N) rational function of s, $r_N^{(k)}(s,\theta)$ is of type (kN, kN).

From Proposition 3.2 of Provoost and Michiels (2024) we have the closed form expression

$$\rho_{k,N}(s,\theta) = \frac{\sum_{j=0}^{N} \phi_{k,N}^{(N-j)}(\theta)\, s^k}{\sum_{j=0}^{N} \phi_{k,N}^{(N-j)}(0)\, s^k},$$

where $\phi^{(j)}$ is the jth derivative of ϕ. Hence, we also have a closed form expression for all $r_N^{(k)}(s,\theta)$. From this and the discussed structure of $r_N^{(k)}$, we straightforwardly see, for symmetric basis functions $\phi_{k,N}$,

$$|r_N^{(k)}(i\omega, -\tau_k)| = 1, \quad \forall \omega \in \mathbb{R}, \quad k = 1, \ldots, m.$$

Note that, with the exception of $r_N^{(1)}(s,0) = 1$, all $r_N^{(k)}$ in G_N^{spl} are only evaluated with θ in their left end point, i.e. $\theta = -\tau_k$. This, together with the above result, then means that for symmetric basis functions, the rational approximants in G_N^{spl} have a better qualitative match with the exponential—in the sense that their magnitudes match—than if using a non-symmetric set of $\phi_{k,N}$ or a single polynomial for multiple delay intervals.

3.2 Descriptor-system reduction for commensurate delays

We limit our attention to the case where the delays are commensurate, that is $\tau_k = d_k \tau$ for some $\tau > 0$ and $d_k \in \mathbb{N}$, with $k = 1, \ldots, m$. In this setting, results for single delay systems can often be extended to multiple delays by using a descriptor-system reformulation of (1) (see e.g. Unger, 2020, Remark 1.2), namely

$$\begin{cases} \dot{\mathbf{x}}_1(t) = A_0 \mathbf{x}_1(t) + \sum_{k=1}^{m} A_k \mathbf{x}_{d_k}(t-\tau) + B\mathbf{u}(t), \\ \mathbf{x}_2(t) = \mathbf{x}_1(t-\tau), \\ \mathbf{x}_3(t) = \mathbf{x}_2(t-\tau), \\ \qquad \vdots \\ \mathbf{x}_{d_m}(t) = \mathbf{x}_{d_m-1}(t-\tau), \end{cases}$$

$$\mathbf{y}(t) = C\mathbf{x}_1(t).$$

We can discretize this system using a slight modification of the polynomial Lanczos tau approach of section 2, by allowing a singular matrix on the left-hand side, yielding the differential algebraic equation:

$$\begin{pmatrix} \varepsilon_0^{(1)} \\ \mathcal{T}_{\phi_N} \\ (\mathbf{0})_{j=1}^{d_m-1} \end{pmatrix} \dot{\xi}_{tN} = \begin{pmatrix} A_0 \varepsilon_0^{(1)} + \sum_{k=1}^{m} A_k \varepsilon_{-\tau}^{(d_k)} \\ \mathcal{D} \\ \left(\varepsilon_{-\tau}^{(j)} - \varepsilon_0^{(j+1)}\right)_{j=1}^{d_m-1} \end{pmatrix} \xi_{tN} + \begin{pmatrix} B \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \mathbf{u}(t),$$

$$\mathbf{y}_N(t) = C\varepsilon_0^{(1)} \xi_{tN},$$

where, unlike equation (5), ξ_{tN} now maps $[-\tau, 0]$ to $\mathbb{C}^{d_m n}$ and $\varepsilon_\theta^{(j)} \xi_{tN}$ selects the subvector from $\xi_{tN}(\theta)$ which corresponds to $\mathbf{x}_j$.

This slight abuse of notation helps to crystallize the connection between this discretization of a descriptor-system and the spline discretization (5). Note first that we can regroup $\xi_{tN} : [-\tau, 0] \to \mathbb{C}^{d_m n}$ into d_m polynomials $\xi_{tN}'^{(j)} : [-\tau, 0] \to \mathbb{C}^n$, each satisfying $\mathcal{T}_{\phi_N} \dot{\xi}_{tN}'^{(j)} = \mathcal{D}\xi_{tN}'^{(j)}$. Through a change of variables we can also think of these as the polynomials $\xi_{tN}^{(j)} : [-j\tau, -(j-1)\tau] \to \mathbb{C}^n$, which, together with the algebraic constraints, means that they form a continuous spline Ξ_{tN} from $[-d_m\tau, 0] = [-\tau_m, 0]$ to $\mathbb{C}^n$. Finally, turning the remaining DAE into an ODE by differentiating the algebraic constraints, we recover a version of (5) using the delays $\{0, \tau, 2\tau, \ldots, d_m\tau\}$ instead of $\{\tau_0, \ldots, \tau_m\}$, using $\mathbf{0}_{n\times n}$ as coefficient matrix where needed.

If $d_k = k$, the resulting discretization is hence a direct equivalent DAE formulation of (5), else the connection of course still holds, but the descriptor-system reformulation will result in a larger discretization due to the creation of additional intervals. Interestingly, these additional intervals result in a system with equidistant delays, informing a straightforward way to attain super-geometric convergence for commensurate delays, as discussed earlier in this section.

4. CONCLUSIONS

We showed that we can partially extend the reported super-geometric convergence of the H^2-norm of a single delay system to multiple discrete delays, by extending the Lanczos tau method to use a spline with the delays as break points in section 3. We see significantly accelerated convergence for general multiple discrete delays and super-geometric convergence when these delays are equidistant, both improvements over the third order convergence reported in Vanbiervliet et al. (2011) for spectral discretizations using a single polynomial.

In the frequency domain, we showed that this corresponds to using distinct rational approximants of the exponential functions for each non-zero delay. This is unlike the Lanczos tau method with a single polynomial where all approximants share the same poles and only the zeroes change with the delay. Whilst not a proof of the accelerated convergence, this does provide some intuition as to why this faster convergence might appear.

In subsection 3.1 a closed form expression for these rational approximations is derived. From the structure of these rational functions, further evidence is given for the accelerated convergence; the modulus along the imaginary axis turns out to be one in the delays, which matches that of the exponential.

Finally, we detailed a connection between the single delay descriptor-system reformulation of systems with commensurate delays and the spline discretization. This connection informs a way to attain super-geometric convergence for commensurate delays by rewriting the system to one with equidistant delays, at the cost of a larger discretization.

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