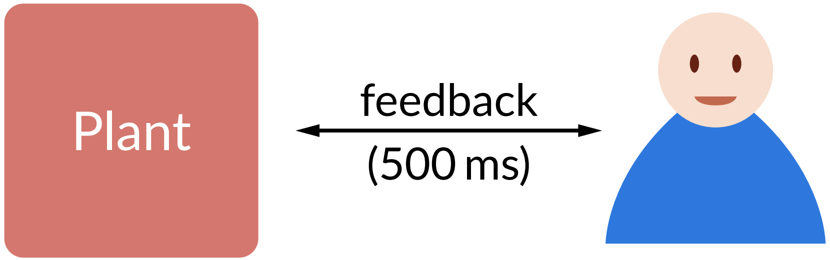
KU LEUVEN

Exploring the Basins of Attraction of Dynamical Systems with Delay

Evert Provoost (KU Leuven)   Francesca Scarabel (Leeds)

Why delay?



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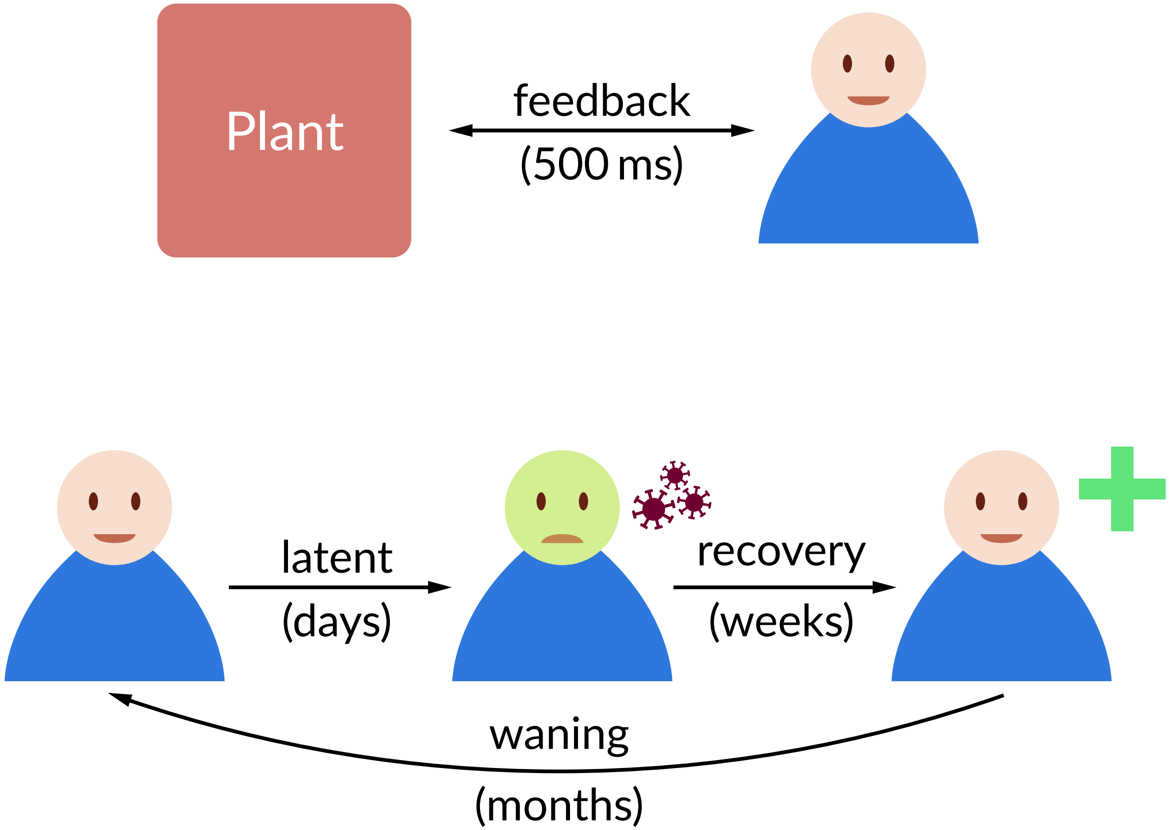
Plant
feedback
(500 ms)
latent
(days)
recovery
(weeks)
waning
(months)

Nicholson's blowflies model with Allee effect

$$\dot{u}(t) = -u(t) + \beta u(t-\tau)^k e^{-u(t-\tau)}$$

[Chang & Shi, 2022, *Discrete Contin Dyn Syst - Ser B*, 27/8: 4551–72]

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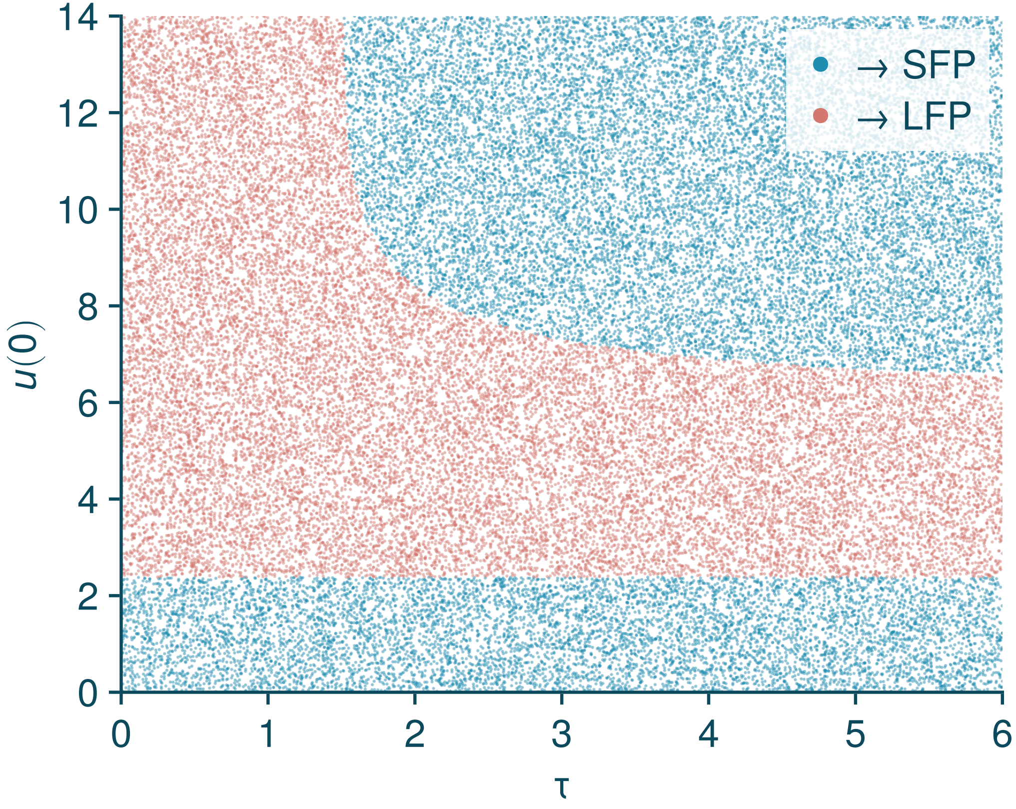

where
$\beta = 0.8$
$k = 4$

[Chang & Shi, 2022, *Discrete Contin Dyn Syst - Ser B*, 27/8: 4551–72]

SIR model with waning and boosting of immunity

$$\dot{S}(t) = \mu - \mu S(t) - \beta I(t)S(t) + e^{-\mu\tau - \nu\beta \int_{t-\tau}^{t} I(u)\,\mathrm{d}u} \left(\gamma I(t-\tau) + \nu\beta R(t-\tau I(t-\tau))\right)$$

$$\dot{I}(t) = \beta I(t)S(t) - (\gamma + \mu)I(t)$$

$$R(t) = 1 - S(t) - I(t)$$

[Barbarossa et al., 2017, *SIAM J Appl Math*, 77/3: 905–23]

SIR model with waning and boosting of immunity

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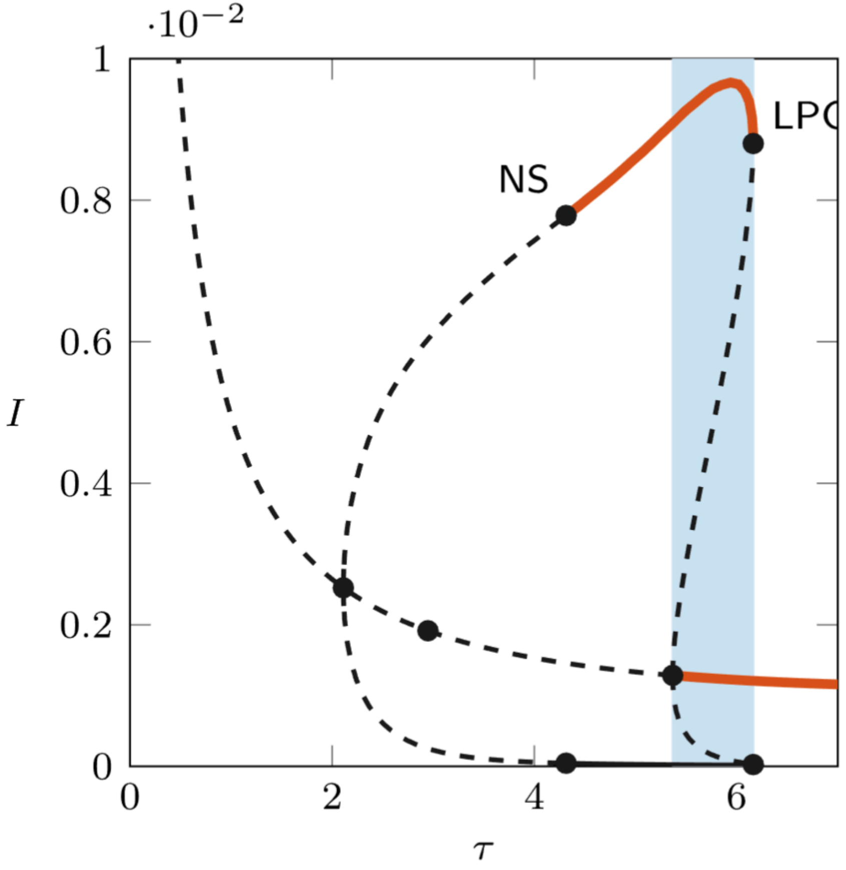

where
$\nu = 3.2$
$\mu = 0.02$
$\beta = 255.3$
$\gamma = 17$
$\tau = 5.6$

Figure from [Scarabel et al., 2025, *J Math Biol*, 91: 30]

Mackey–Glass equation

$$\dot{u}(t) = -\gamma u(t) + \beta f(u(t - \tau)) \quad \text{with} \quad f(x) = \frac{\theta^n x}{\theta^n + x^n}$$

[Mackey & Glass, 1977, *Science*, 197/4300: 287–9]

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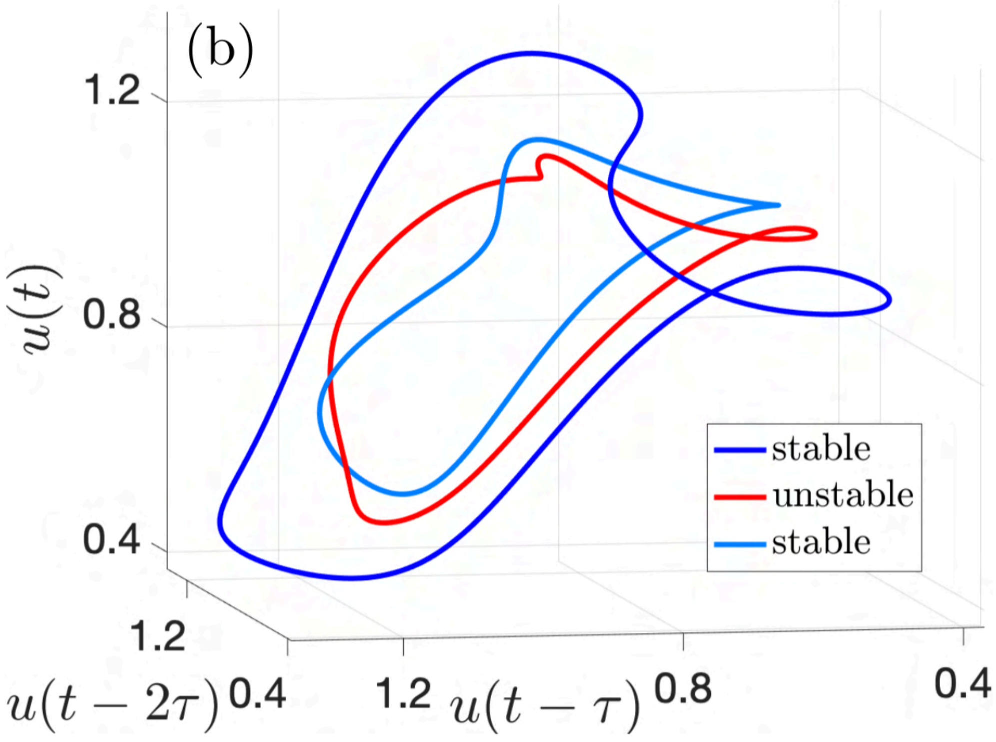

where
$n = 14.5$
$\tau = 1.3$
$\gamma = 1$
$\theta = 1$
$\beta = 2$

Figure from [Duruisseaux & Humphries, 2022, *J Comput Dyn*, 9/3: 421–50]

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[Stender & Hoffmann, 2022, *Nonlinear Dyn*, 107/2: 1451–68]

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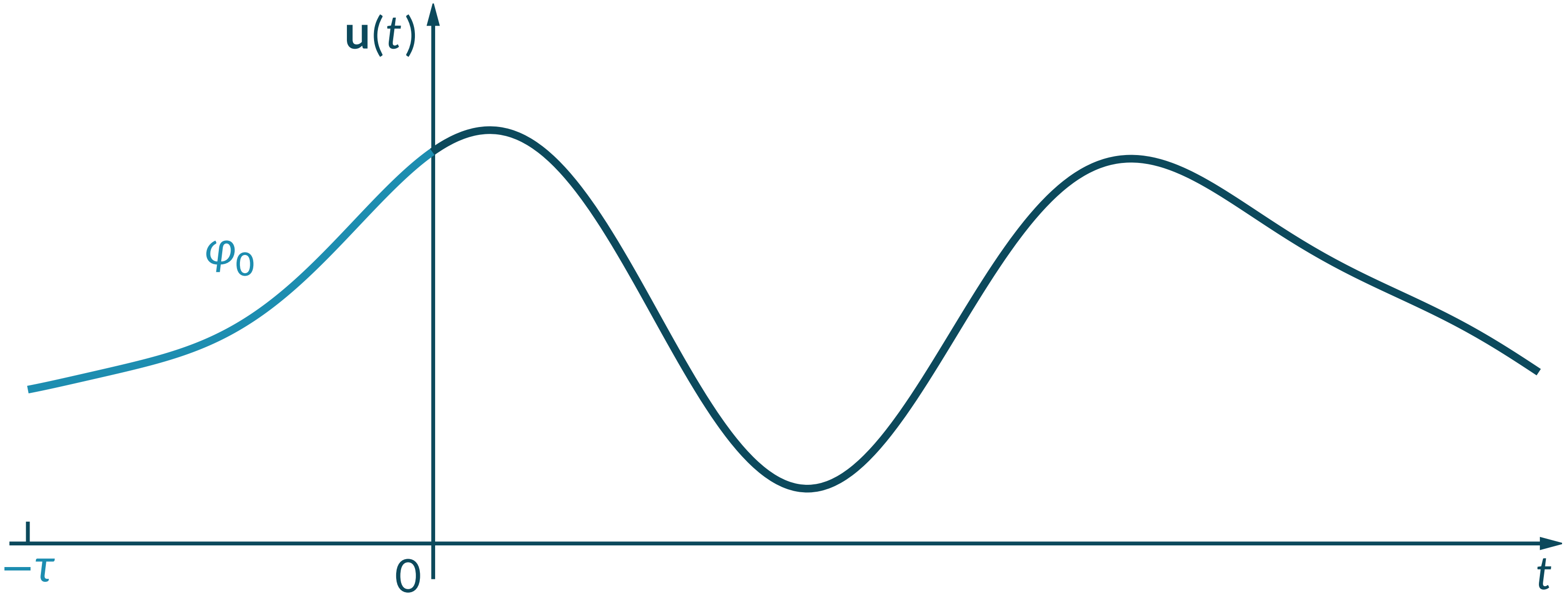
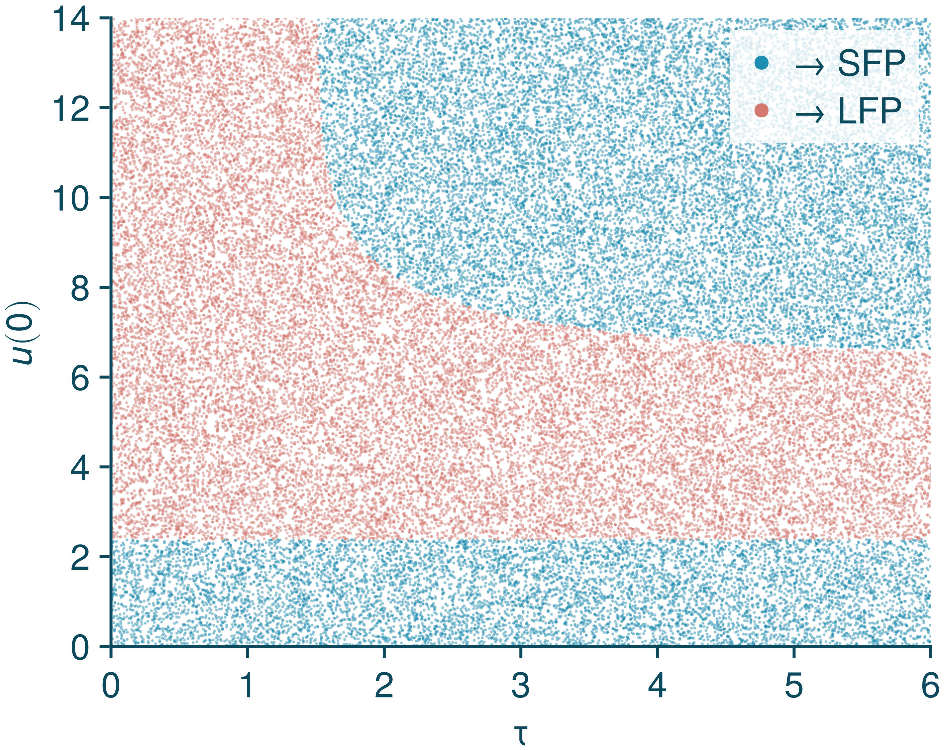
For ODEs one can just simulate random initial conditions.

Can we do the same for DDEs?

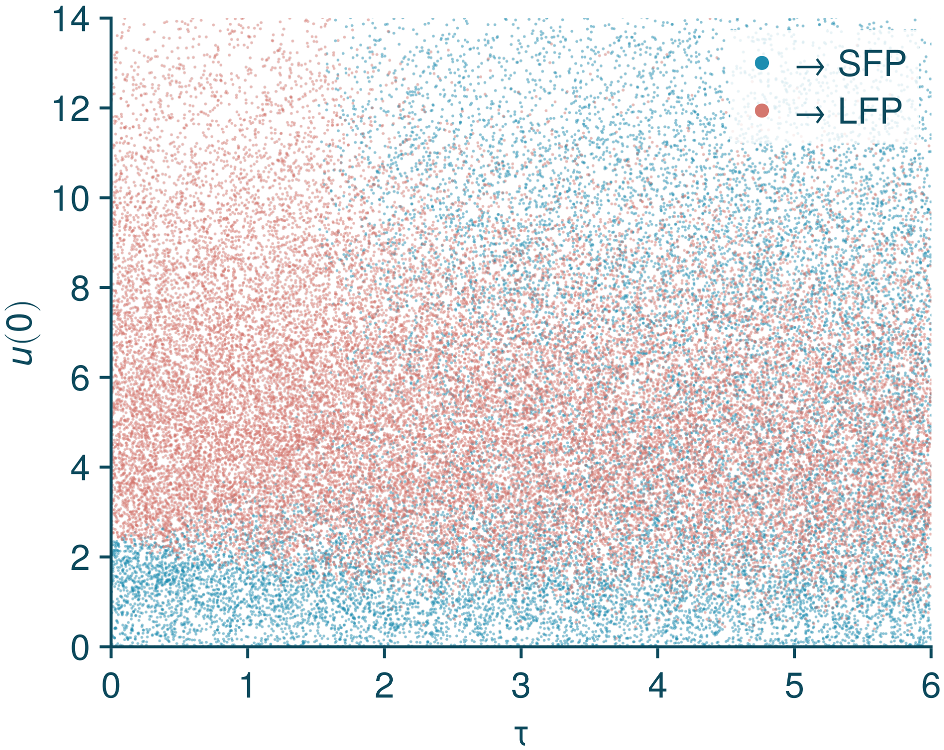
DDEs are infinite dimensional systems...



…and that matters



constant φ_0



variable φ_0

How to sample initial conditions?

Which attractors are there?

How to interpret the resulting basins?

How to tackle sampling?

We assume:

- well-behaved functions,[*]
- with finite support,
- a typical value range.

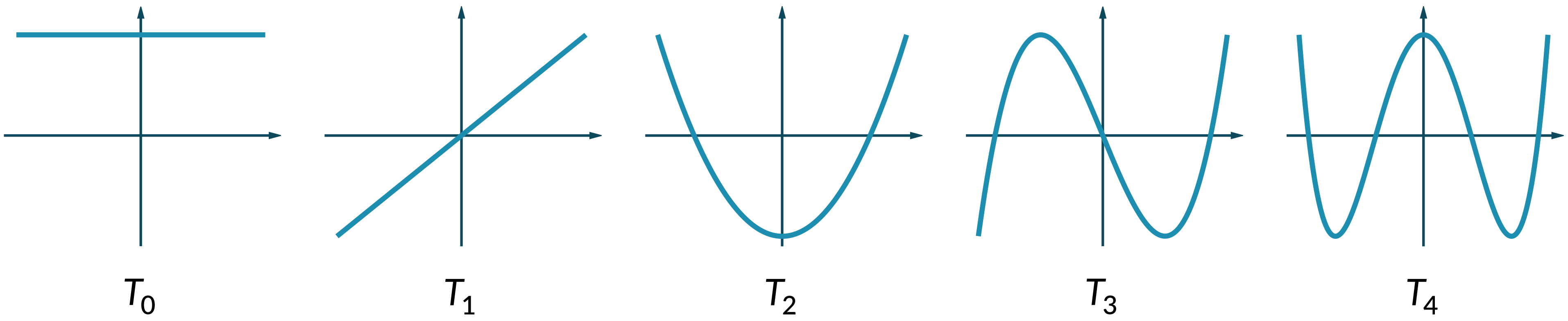
We want:

- finite-dimensional,
- ‘linear’ structure,
- truncatable.

[*] Most DDEs are smoothing.

Chebyshev series

$$p(\theta) = \sum_{i=0}^{N} c_i T_i(\theta), \quad \text{where} \quad T_i(\theta) = \cos(i \,\text{acos}(\theta)).$$



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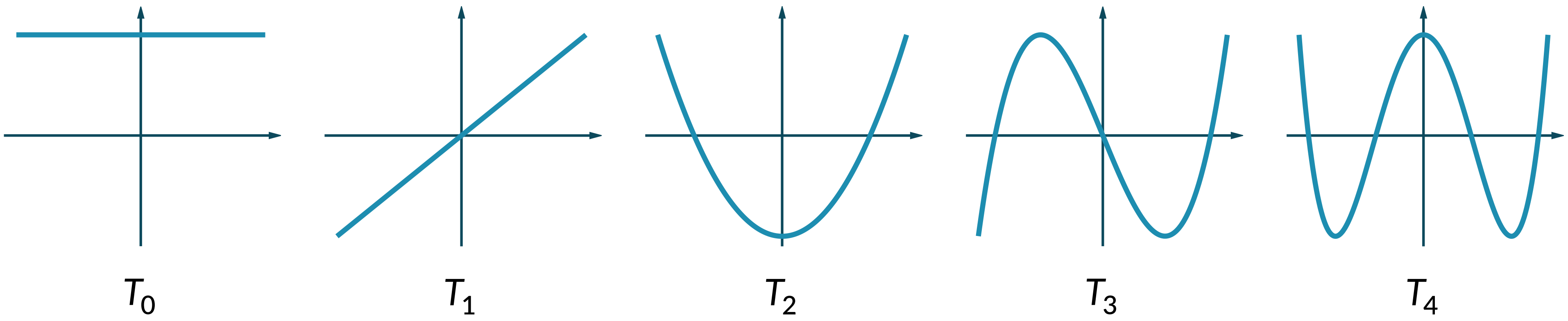

Prop: the series' decay rate relates to the function's regularity.

Sampling initial conditions

Sample $p(\theta) = \sum_{i=0}^{N_h} c_i T_i(\theta)$ with

$$c_0 \sim N(\mu_0, \sigma_0) \quad \text{and} \quad c_i \sim N\left(0, \sigma\rho^i\right).$$

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If required to be positive, compute x_k s.t.

$$\varphi_0(\theta) = \sum_{k=0}^{2N_h} x_k T_k(\theta) = p(\theta)^2.$$

Squaring is fast using FFT [Ahmed & Fisher, 1970, *Int J Comput Math*, 2/1–4: 307–17].

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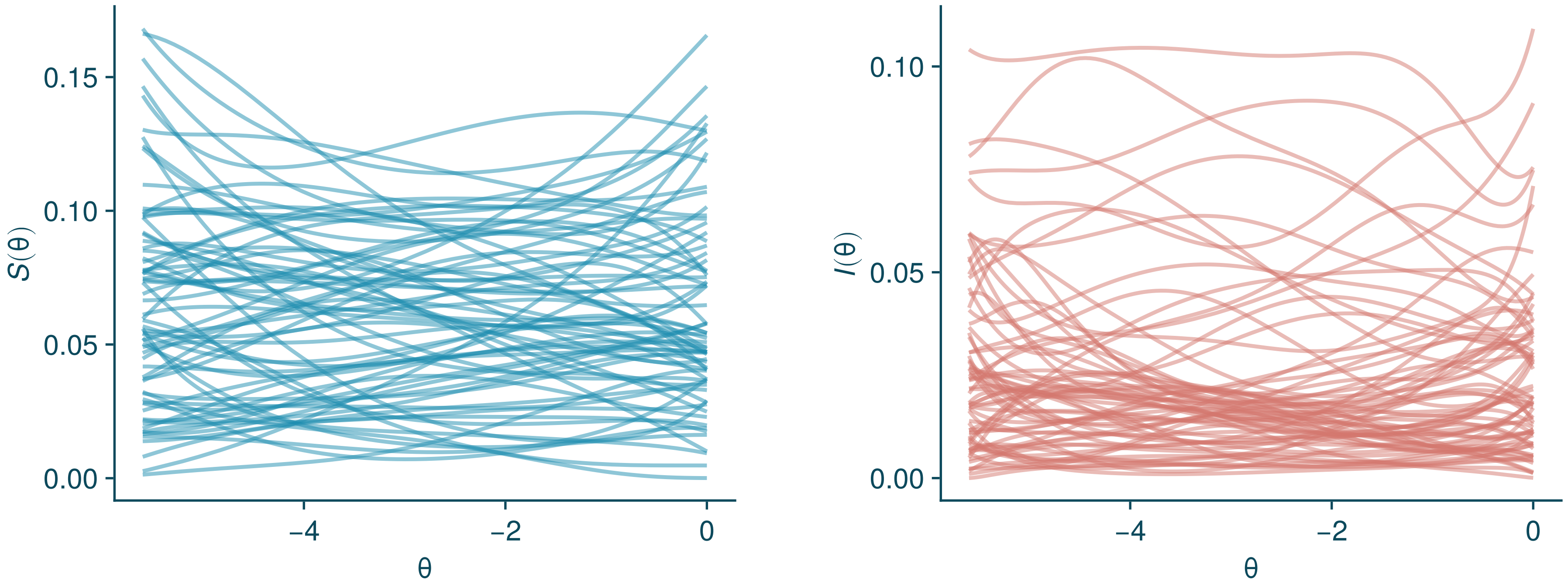
Repeat for every variable, yielding the initial condition $\boldsymbol{\varphi}_0$.

Sampling for the SIR model

For S we use $\mu_0 = \sigma_0 = 0.06$, $\sigma = 0.1$, and $\rho = 0.4$,
for I we use $\mu_0 = 0.02$, $\sigma = 0.05$, and $\rho = 0.6$, both with $N_h = 8$.



Detecting attractors

As trajectory captures state, the `bSTAB` approach of ODEs works:

[Stender & Hoffmann, 2022, *Nonlinear Dyn*, 107/2: 1451–68]

Generate n initial conditions $\boldsymbol{\varphi}_{0,i}(\theta)$.
for $i = 0, \ldots, n$ **do**
    Compute trajectory $\boldsymbol{u}_i(t)$ starting from $\boldsymbol{\varphi}_{0,i}(\theta)$.
    Extract feature vector from steady-state $\boldsymbol{X}_i = f\left(\boldsymbol{u}_i(t > t^*)\right)$.
end
Cluster feature vectors $\boldsymbol{X}_i$ using, e.g., k-means clustering.

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```

Use some clustering quality metric to validate result, e.g. the silhouette score. [Rousseeuw, 1987, *J Comput Appl Math*, 20: 53–65]

Attractors of the models

We use the maximum and minimum as feature vector and $n = 1000$.

Blowflies

1. $u_{\text{min}} \approx u_{\text{max}} \approx 0$
2. $u_{\text{min}} \approx u_{\text{max}} \approx 3.71$

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SIR model

1. $I_{\min} \approx I_{\max} \approx 1.2 \times 10^{-3}$
2. $I_{\min} \approx 0$, $I_{\max} \approx 9 \times 10^{-3}$
3. $I_{\min} \approx 0$, $I_{\max} \approx 0.49$ (*new*)

score: 0.9999

Selecting important features (FDA)

Find $\boldsymbol{w}_i$ that maximize

$$\lambda_i = \frac{\frac{1}{2}\sum_{c_j \neq c_k} \langle \boldsymbol{w}_i, \boldsymbol{x}_j - \boldsymbol{x}_k \rangle^2}{\frac{1}{2}\sum_{c_j = c_k} \langle \boldsymbol{w}_i, \boldsymbol{x}_j - \boldsymbol{x}_k \rangle^2}$$

and are orthogononal to all $\boldsymbol{w}_j$ for $j < i$.

Selecting important features (LFDA)

Find $\boldsymbol{w}_i$ that maximize

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and are orthogononal to all $\boldsymbol{w}_j$ for $j < i$.

[Sugiyama, 2007, *J Mach Learn Res*, 8/37: 1027–61]

Affinity

Samples $\boldsymbol{x}_j$ and $\boldsymbol{x}_k$ have an affinity

$$a_{jk} = \exp\left(-\frac{\|\boldsymbol{x}_j - \boldsymbol{x}_k\|^2}{\sigma_j \sigma_k}\right),$$

Where σ_j is the distance from $\boldsymbol{x}_j$ to its seventh nearest neighbour.

[Zelnik-Manor & Perona, 2005, *Adv Neural Inf Process Syst 17*, 1601–8]

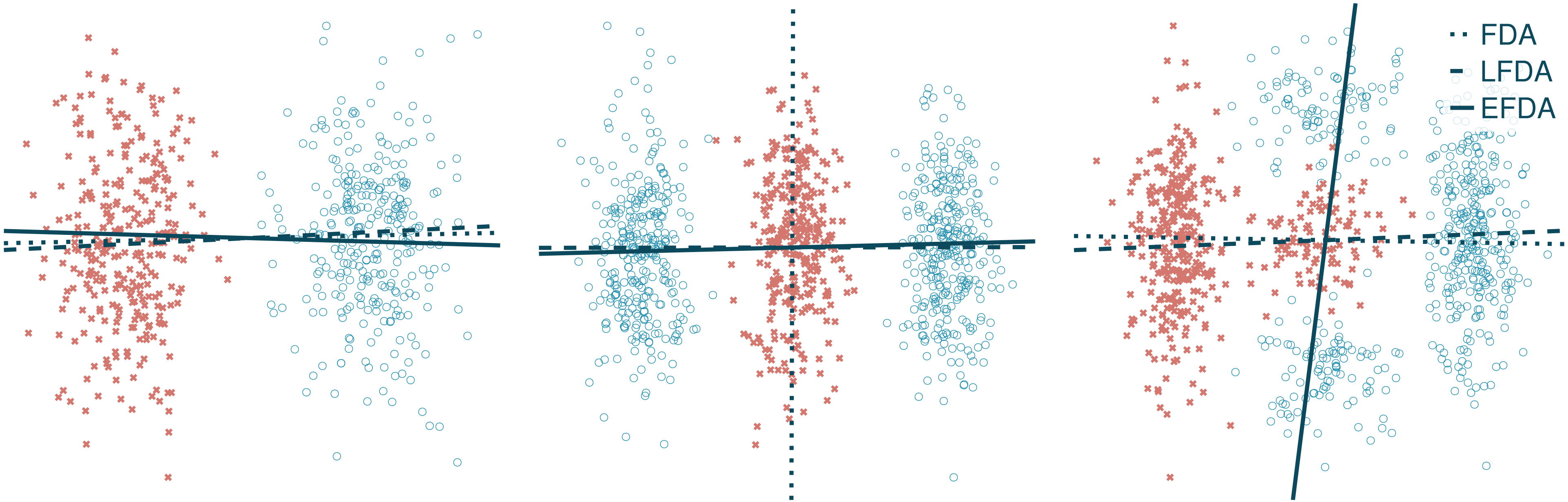
Selecting important features (EFDA)

Find $\boldsymbol{w}_i$ that maximize

$$\lambda_i = \frac{\frac{1}{2}\sum_{c_j \neq c_k} \nu a_{jk} \langle \mathbf{w}_i, \mathbf{x}_j - \mathbf{x}_k \rangle^2}{\frac{1}{2}\sum_{c_j = c_k} a_{jk} \langle \mathbf{w}_i, \mathbf{x}_j - \mathbf{x}_k \rangle^2}$$

and are orthogononal to all $\boldsymbol{w}_j$ for $j < i$.

Selecting important features (comparison)



Finding the optimizer

As the λ_i are Rayleigh quotients, the $\boldsymbol{w}_i$ are the solutions of

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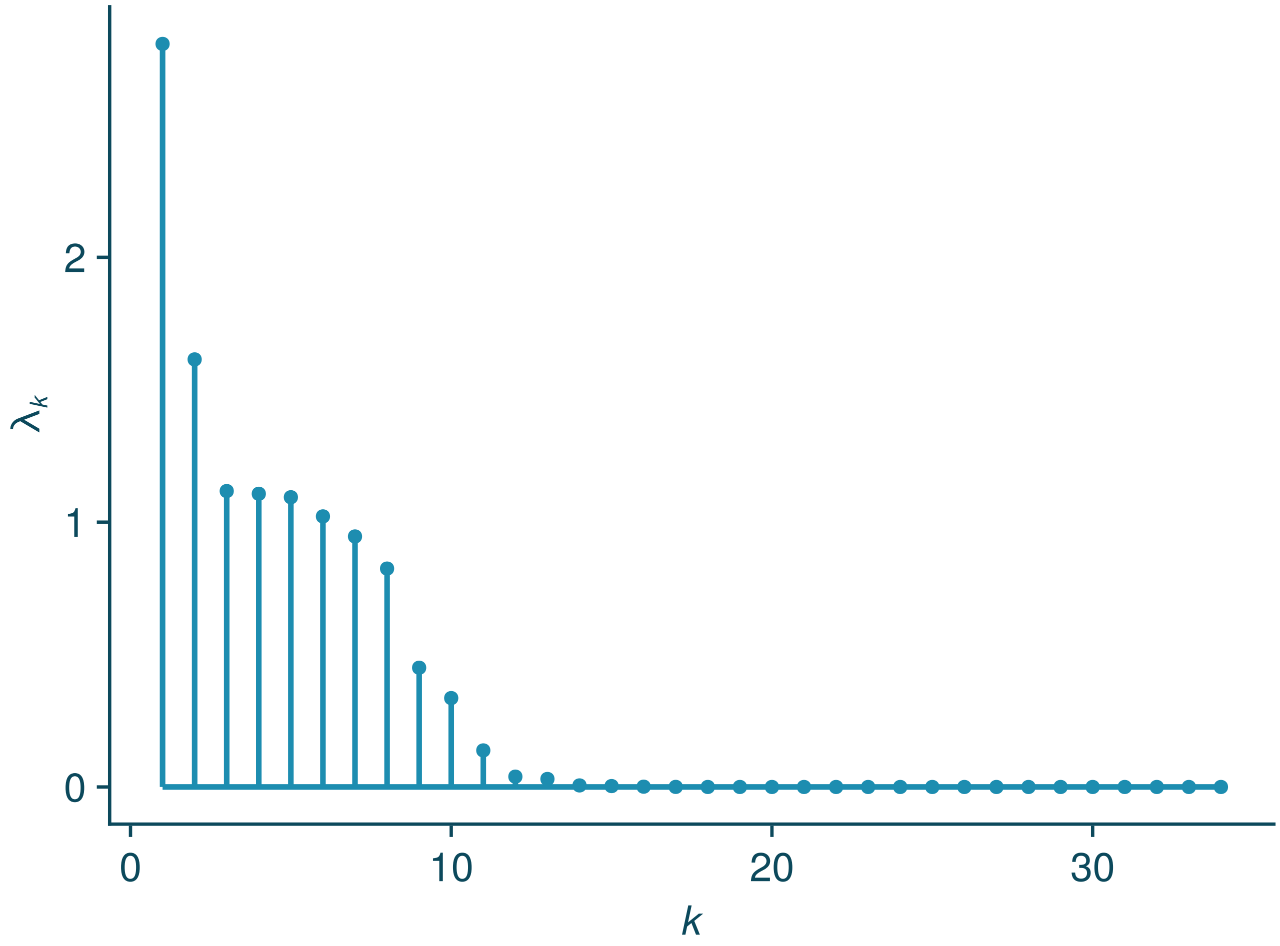
As $n \gg p$, construction of S^w and S^b dominate.

Useful optimization: for large n, take $a_{jk} = 0$ beyond, e.g., the 50^{th} nearest neighbour.

λ_i of SIR model



Interpreting projections

As $\boldsymbol{x}_j$ represent polynomial u_j, the must exist a polynomial f_i of the same degree such that

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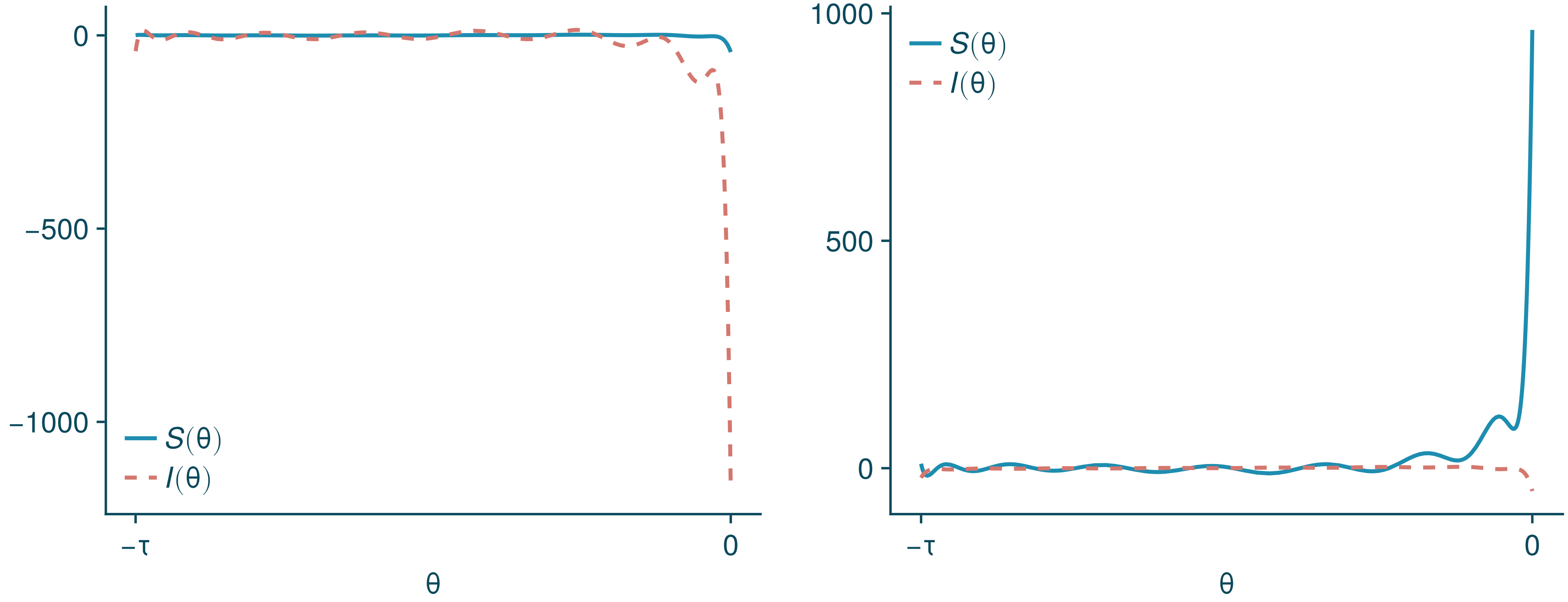
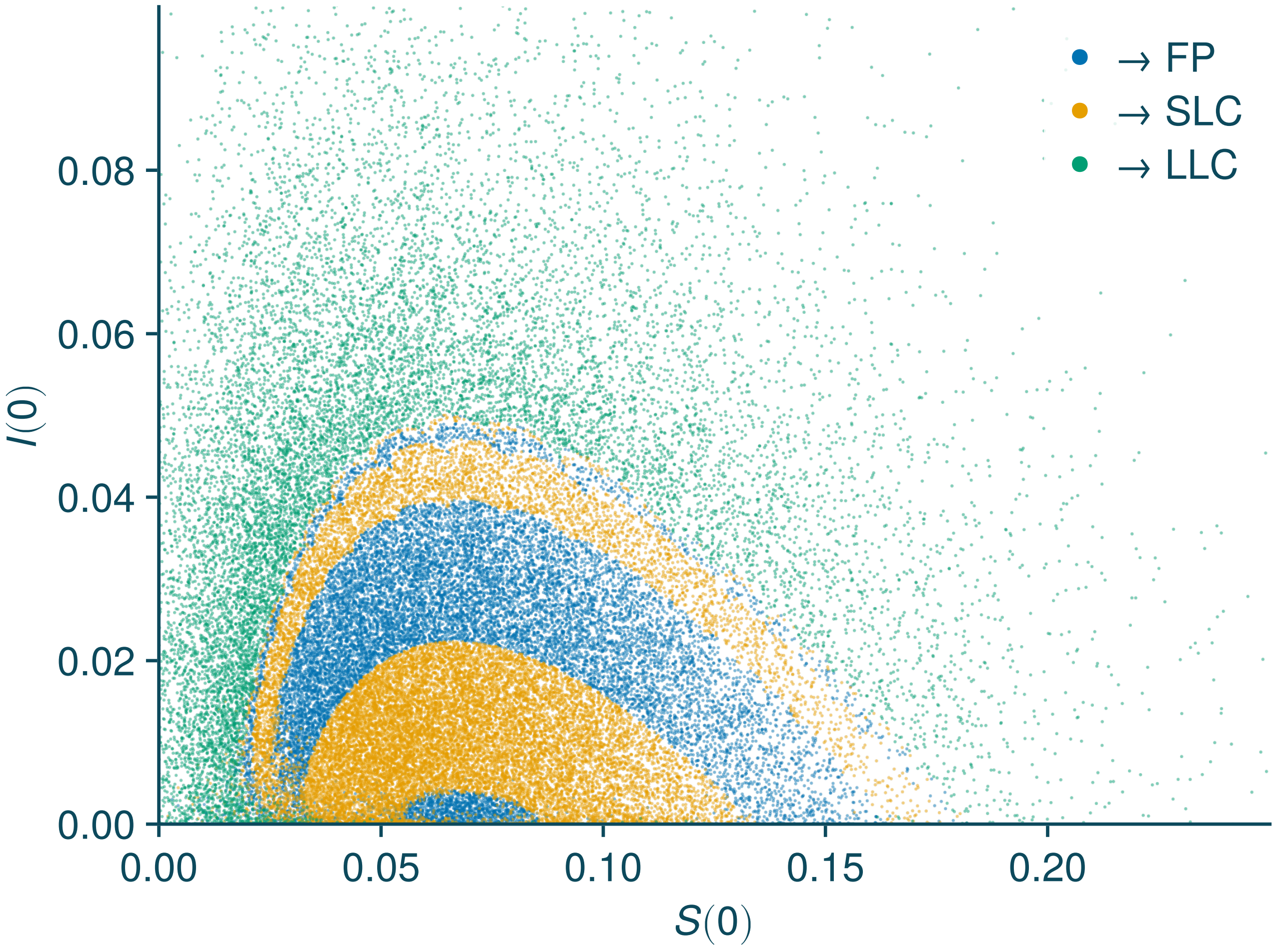
This polynomial is easily found in the Legendre basis.

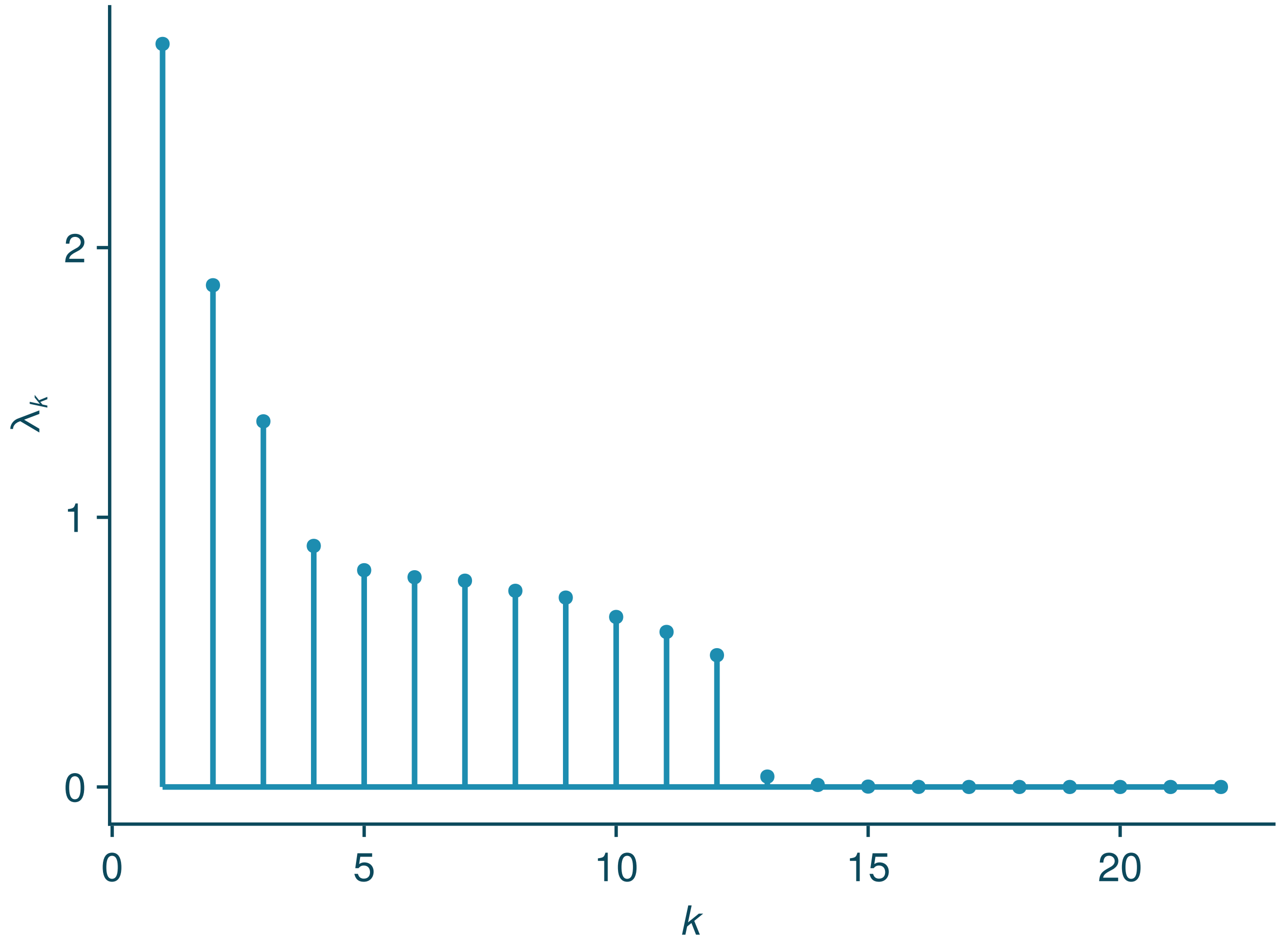
w_i of SIR model



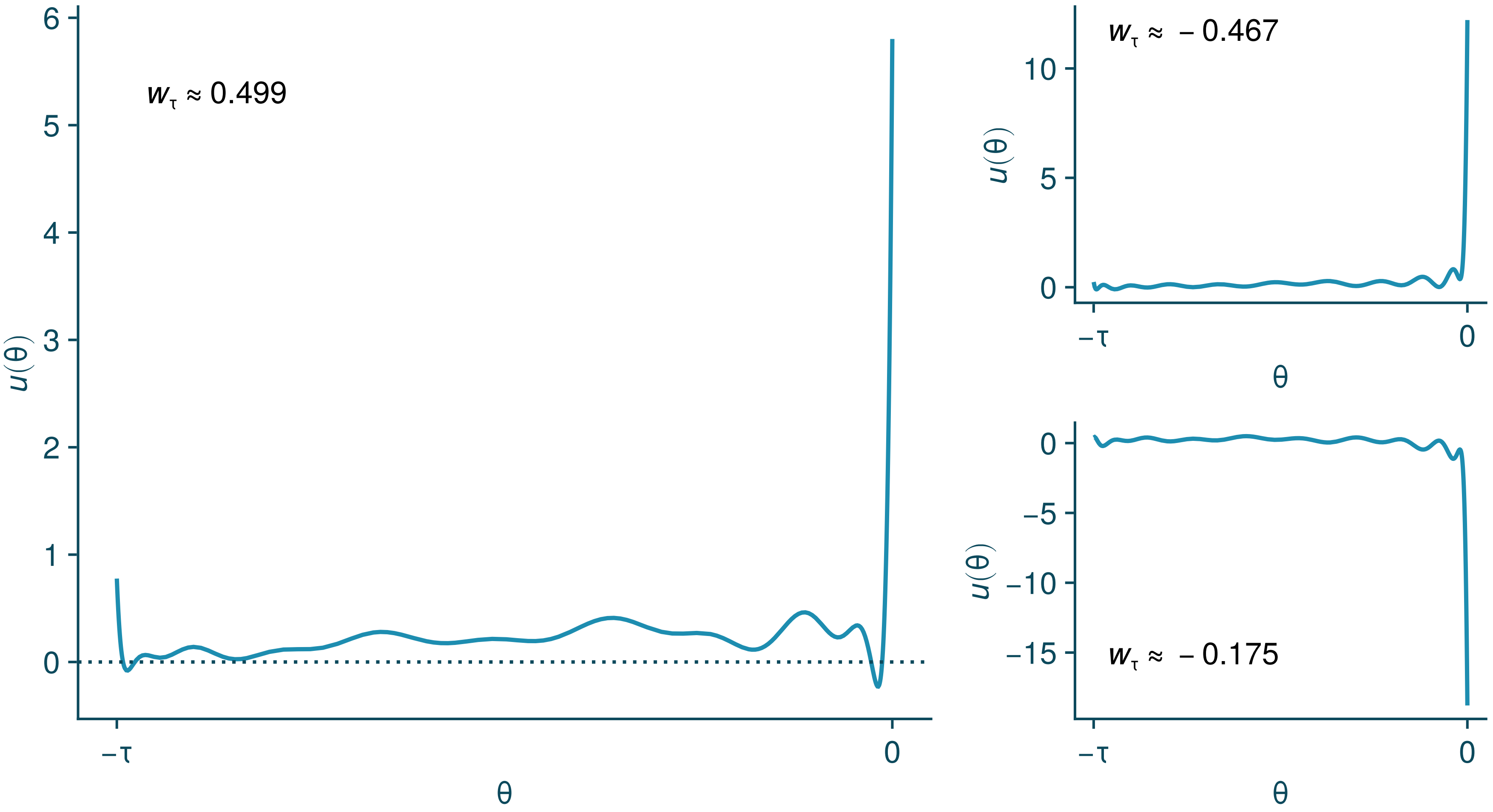
Basins of SIR model



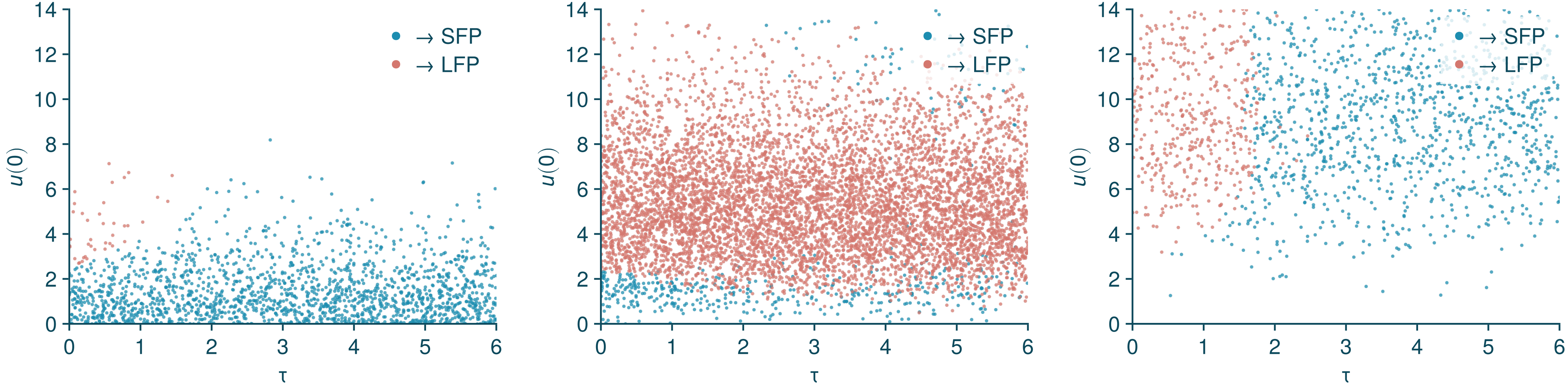
λ_i of blowflies model



w_i of blowflies model

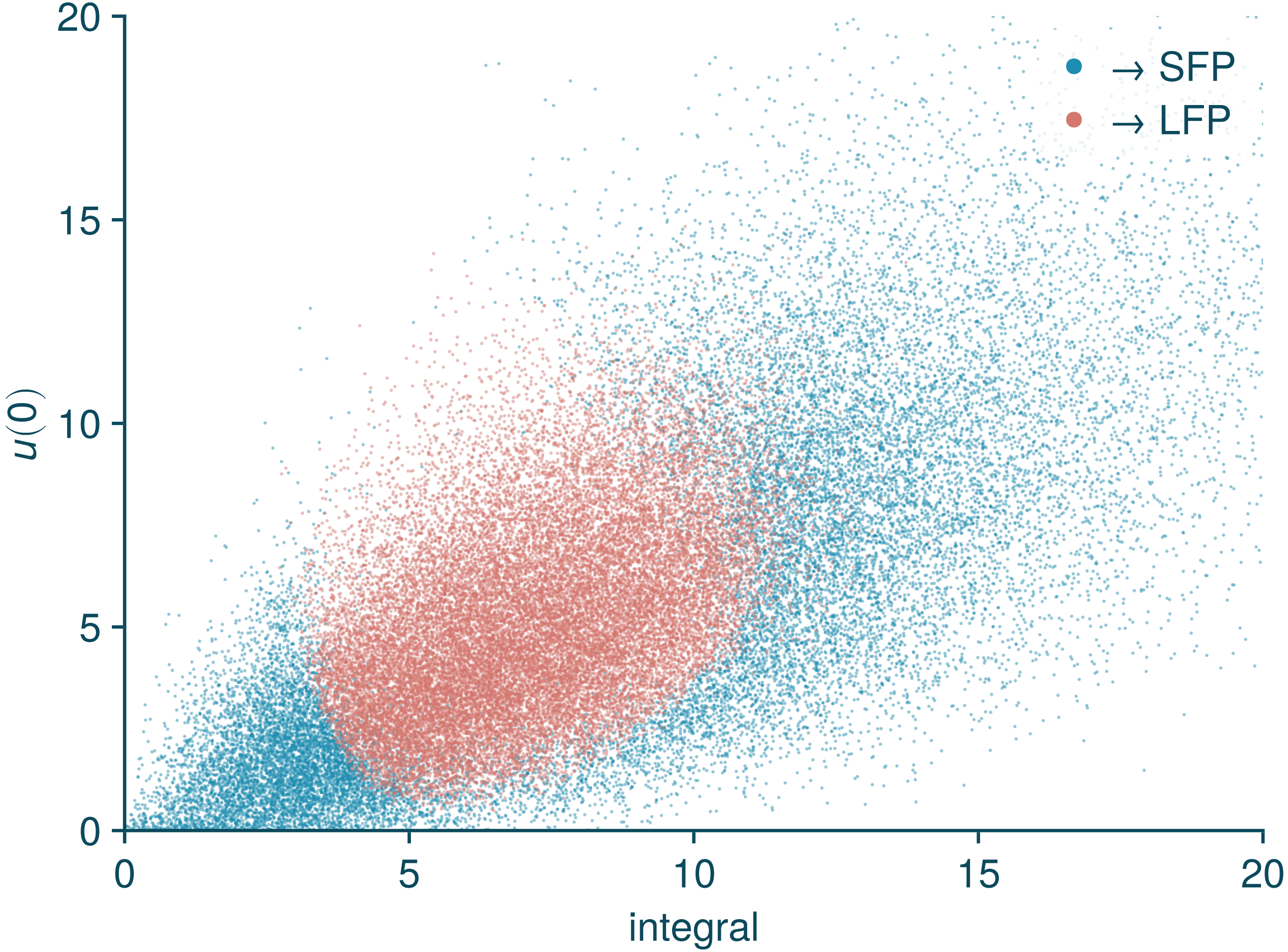


Basins of blowflies model (integral slices)

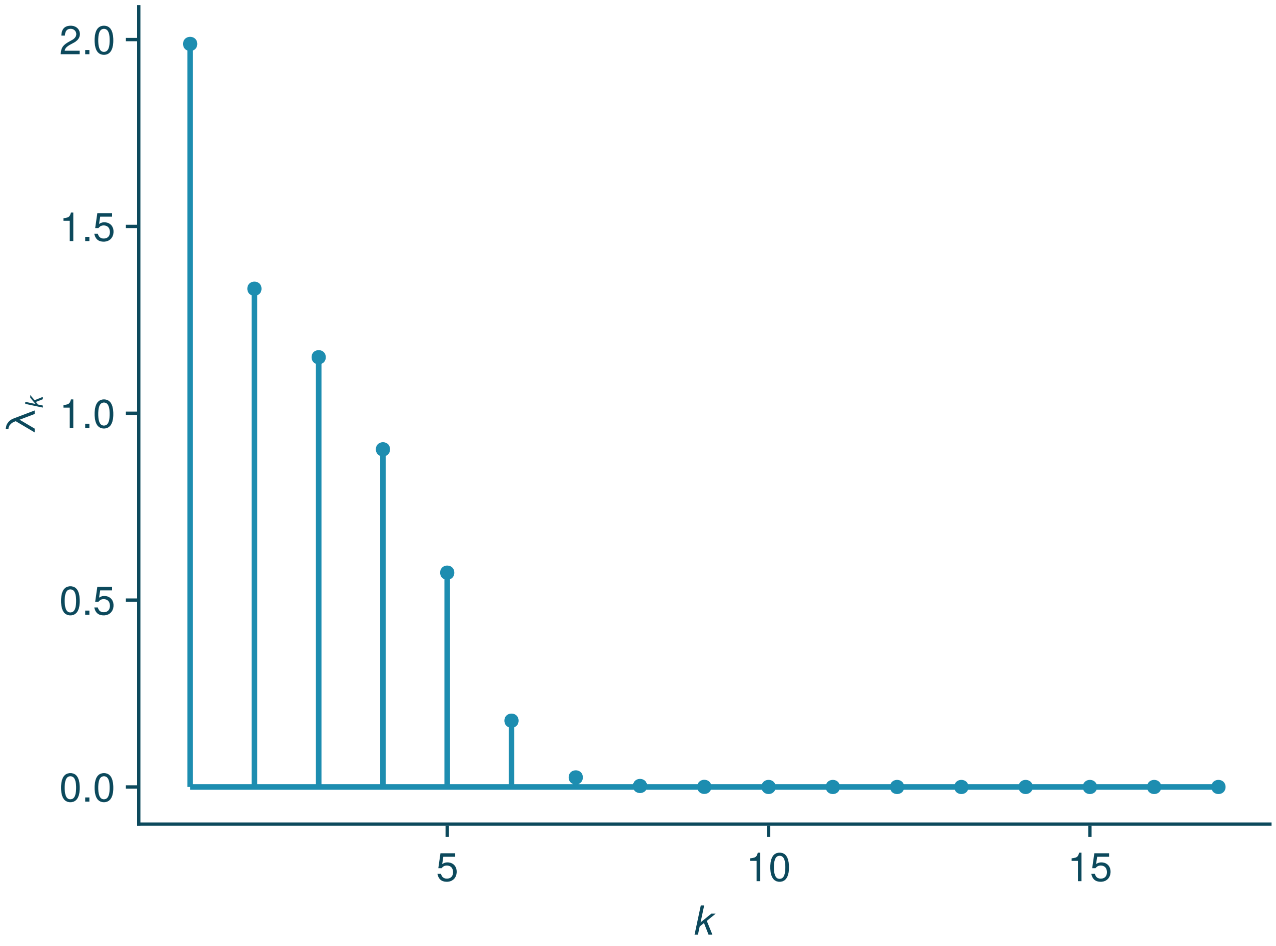


low integral med. integral high integral

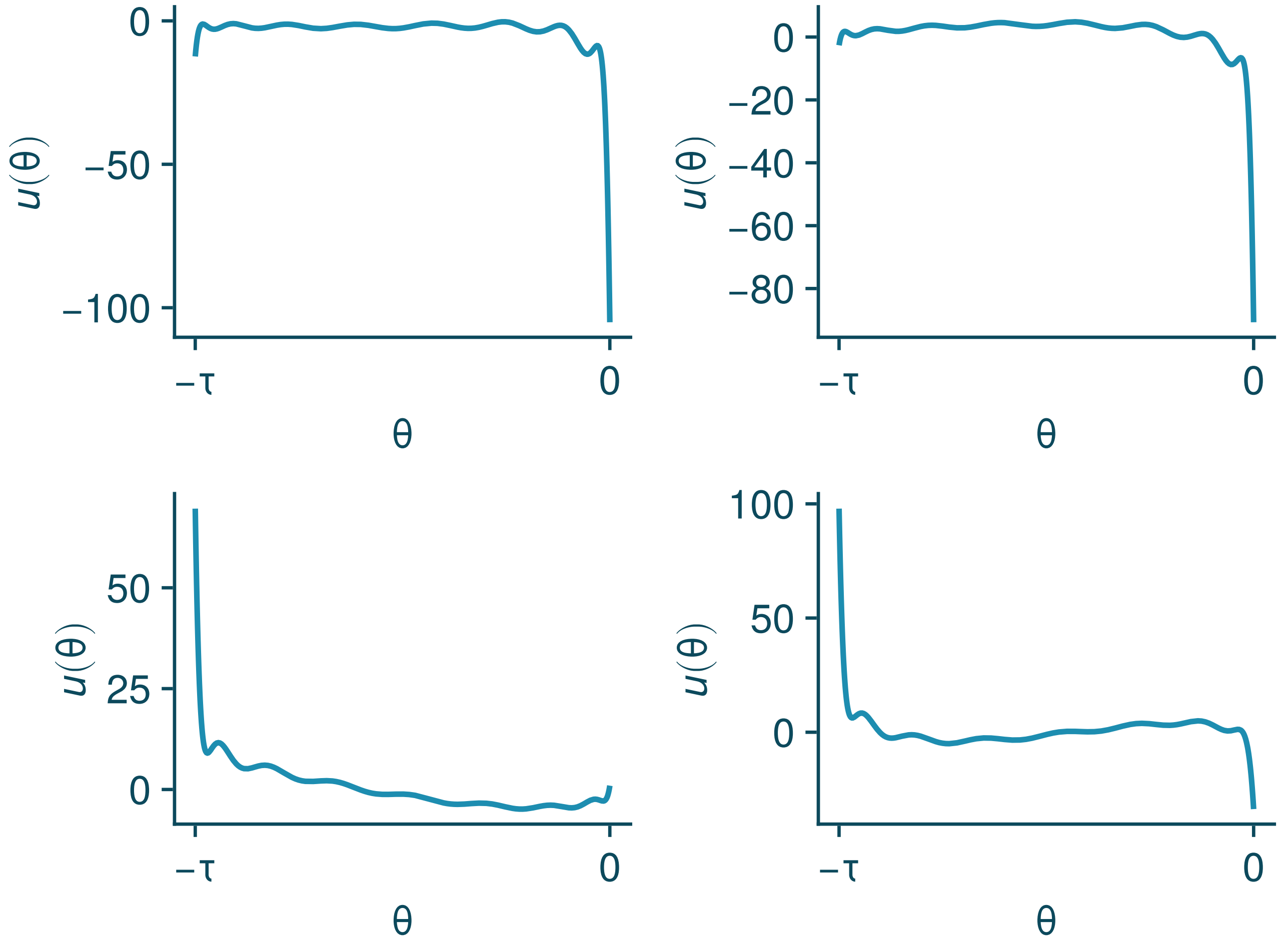
Basins of blowflies model ($\tau = 3$)



λ_i of Mackey–Glass model



w_i of Mackey–Glass model



Contributions

Devised a scheme to sample initial conditions for non-linear DDEs, which can be used to detect attractors.

A dimensionality reduction method to detect deciding features of initial conditions.

We applied these to a number of sample models, recovering known attractors, discovering a new attractor for the SIR model and finding a projection of the initial conditions separating the basins for the blowflies and SIR models.