

Analysing Linear Time-Delay Systems with Lanczos' Tau Method

Evert Provoost and Wim Michiels

Goals

Computing the spectrum:

find $s \in \mathbb{C}$ such that

$$\det(sI_n - \sum_{k=0}^m A_k e^{-\tau_k s}) = 0.$$

Computing the H^2 -norm:

$$\max_{0 < \alpha < \infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \|G(\alpha + i\beta)\|_F^2 d\beta \right)^{\frac{1}{2}}.$$

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Idea: use a delay-free ‘proxy’ model.

Today's model

$$\dot{\mathbf{x}}(t) = \sum_{k=0}^m A_k \mathbf{x}(t - \tau_k) + B\mathbf{u}(t),$$
$$\mathbf{y}(t) = C\mathbf{x}(t).$$

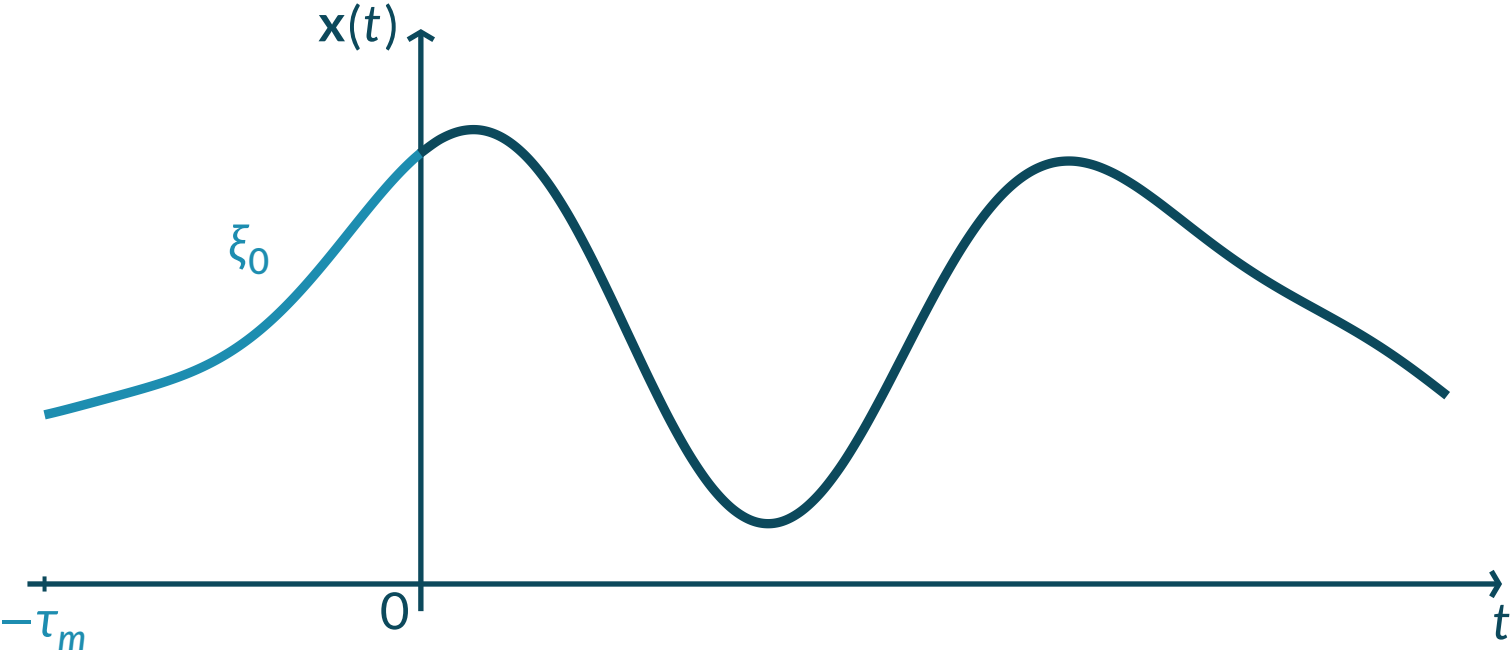
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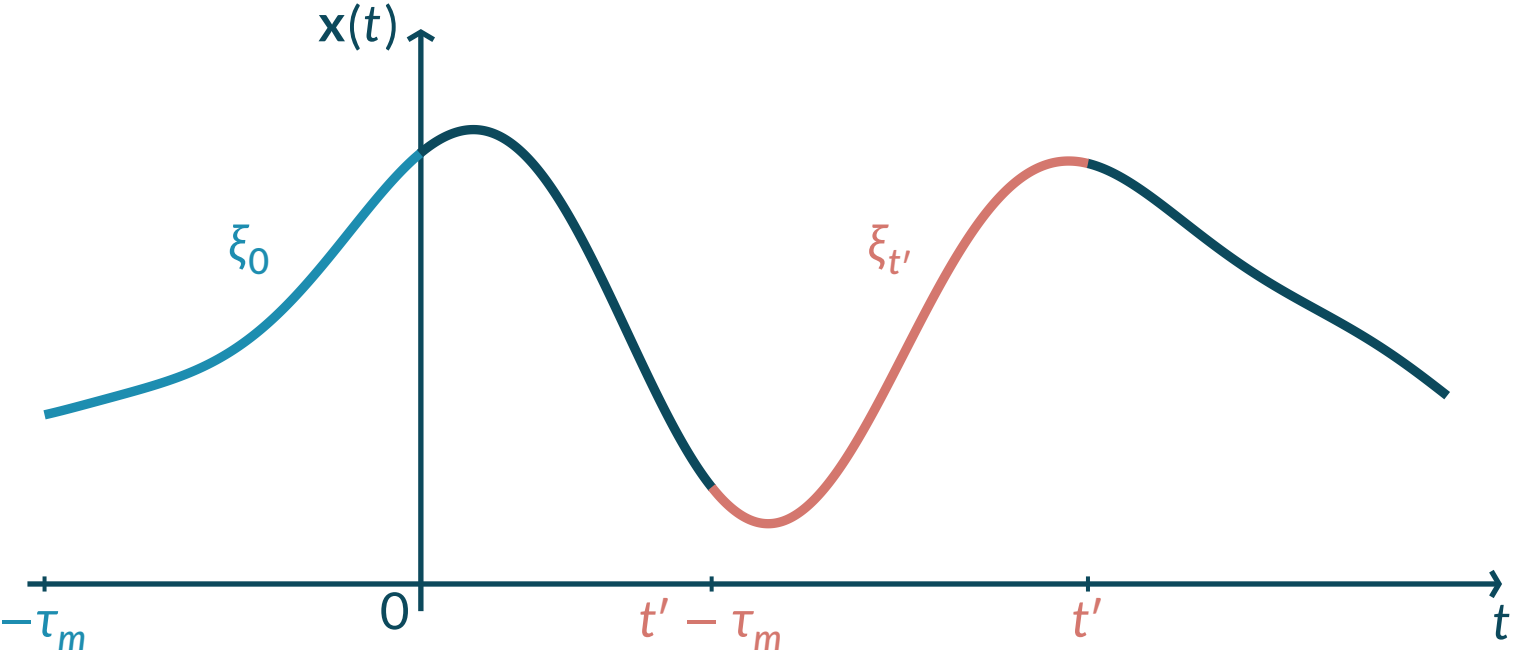
with transfer function

$$G(s) = C(sI_n - \sum_{k=0}^m A_k e^{-\tau_k s})^{-1} B.$$

The advection PDE perspective



The advection PDE perspective



The advection PDE perspective

$$\begin{pmatrix} \varepsilon_0 \\ \text{Id} \end{pmatrix} \dot{\xi}_t = \begin{pmatrix} \sum_{k=0}^m A_k \varepsilon_{-\tau_k} \\ \mathcal{D} \end{pmatrix} \xi_t + \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix} \mathbf{u}(t),$$
$$\mathbf{y}(t) = C \varepsilon_0 \xi_t.$$

The advection PDE perspective → the discretization

$$\begin{pmatrix} \varepsilon_0 \\ \mathcal{T}_{N-1} \end{pmatrix} \dot{\xi}_{tN} = \begin{pmatrix} \sum_{k=0}^m A_k \varepsilon_{-\tau_k} \\ \mathcal{D} \end{pmatrix} \xi_{tN} + \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix} \mathbf{u}(t),$$
$$\mathbf{y}_N(t) = C \varepsilon_0 \xi_{tN},$$

where $\mathcal{T}_{N-1} \xi = \xi - \frac{\langle \xi, \varphi_N \rangle}{\|\varphi_N\|^2} \varphi_N$.

See P. & Michiels (2024) and Ito & Teglas (1986).

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As $\varphi_N(0) \neq 0$, $\ker \varepsilon_0 \cap \ker \mathcal{T}_{N-1}$ is trivial.

See P. & Michiels (2024) and Ito & Teglas (1986).

Stability analysis

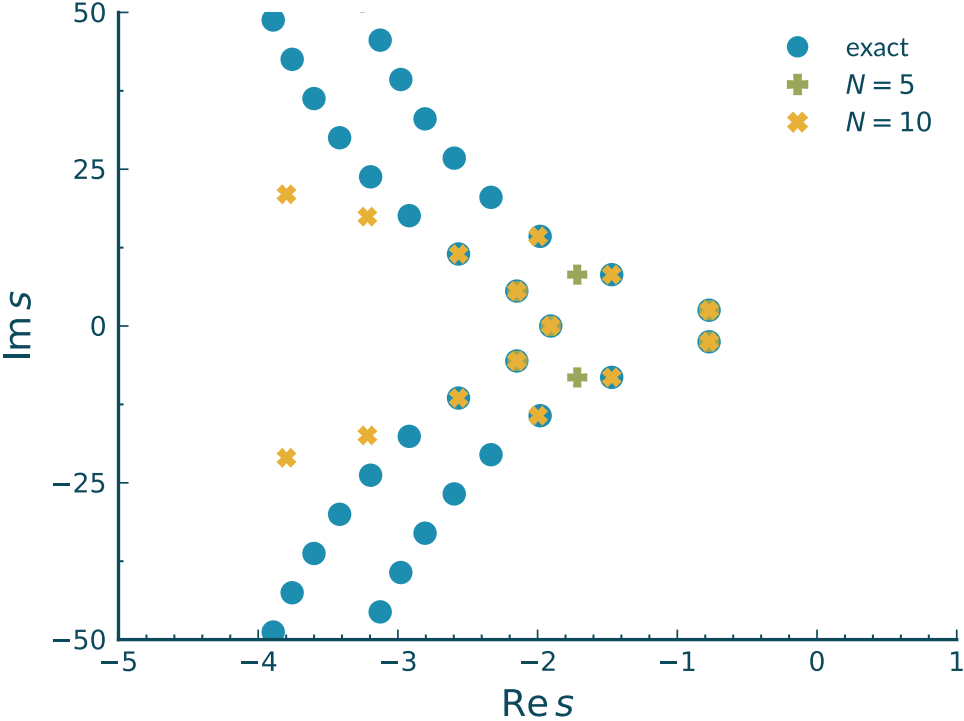
Approximating the characteristic roots

$$\{s \in \mathbb{C} \mid \det(sI_n - \sum_{k=0}^m A_k e^{-\tau_k s}) = 0\} \approx \{s \in \mathbb{C} \mid \det(sE_N - A_N) = 0\}$$

See Ito (1985).

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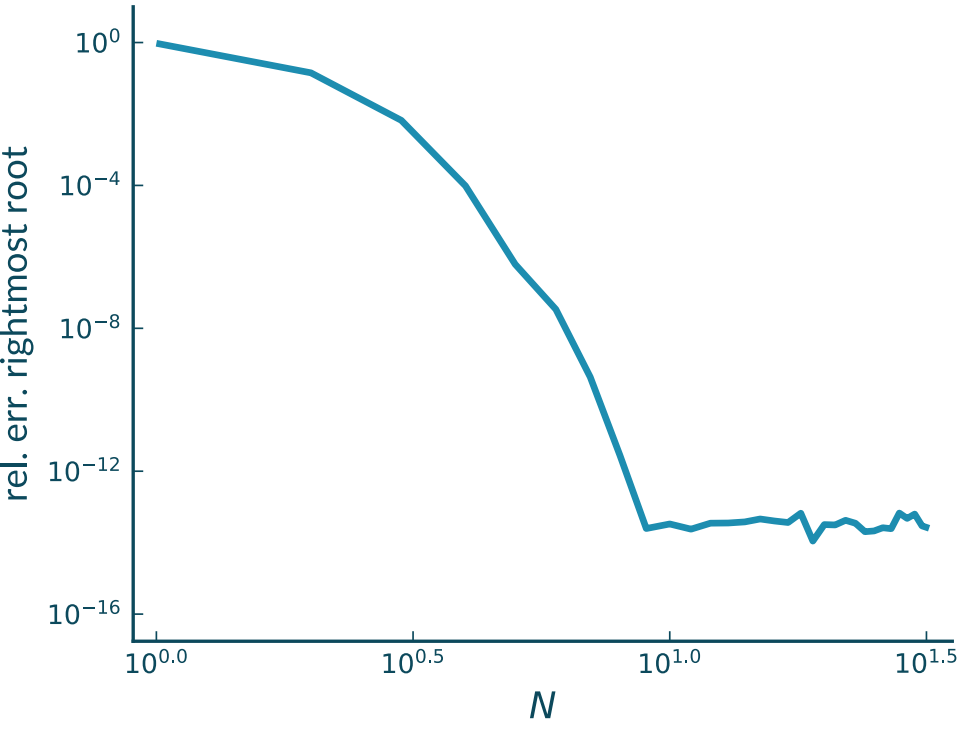
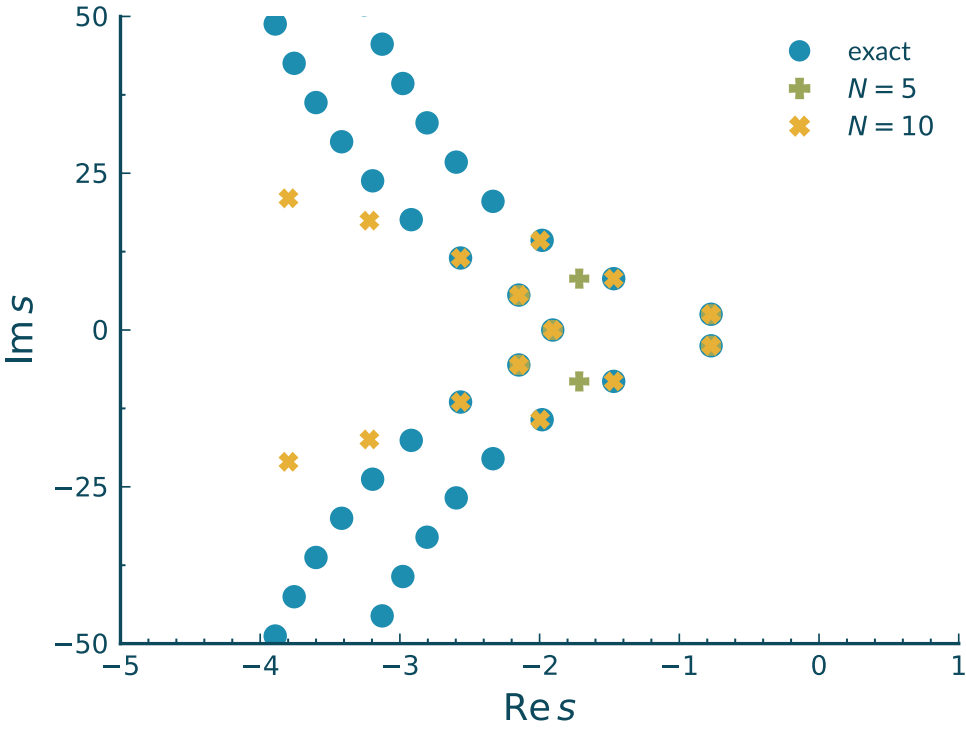
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See Ito (1985).

Connection to collocation

$$\begin{pmatrix} \varepsilon_0 \\ \mathcal{T}_{N-1} \end{pmatrix} \dot{\xi}_{tN} = \begin{pmatrix} \sum_{k=0}^m A_k \varepsilon_{-\tau_k} \\ \mathcal{D} \end{pmatrix} \xi_{tN} + \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix} \mathbf{u}(t),$$
$$\mathbf{y}_N(t) = C \varepsilon_0 \xi_{tN}.$$

Connection to collocation

$$\dot{\xi}_{tN}(\theta) - \mathbf{c}\varphi_N(\theta) = \frac{d}{d\theta}\xi_{tN}(\theta)$$

See Lanczos (1938).

Connection to collocation

$$\dot{\xi}_{tN}(\theta_k) = \frac{d}{d\theta} \xi_{tN}(\theta_k)$$

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We can reuse results, in particular:

φ_N with 'good' zeroes \implies geometric convergence of roots.

See Breda et al. (2015, Theorem 5.1).

Self-nesting

As $\text{span} \{\varphi_k\}_{k=0}^{N_1} \subset \text{span} \{\varphi_k\}_{k=0}^{N_2}$ for $N_1 < N_2 \implies$ trivially self-nesting:

$$E_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} -2 & -2 & -2 & \\ & 2 & 2 & \\ & & 6 & 6 \\ & & & 1 \\ & & & & 14 \end{bmatrix}.$$

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$$E_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} -2 & -2 & -2 & & \\ & 2 & & & 2 \\ & & 6 & & \\ & & & 1 & 1 \\ & & & & 14 \\ & & & & & 18 \end{bmatrix}.$$

Ultraspherical methods

We have $\frac{d}{d\theta} T_k = kU_{k-1}$.

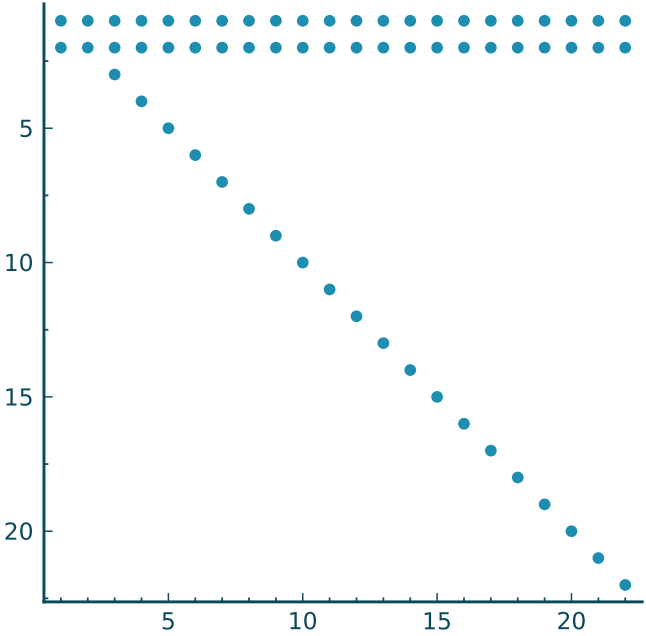
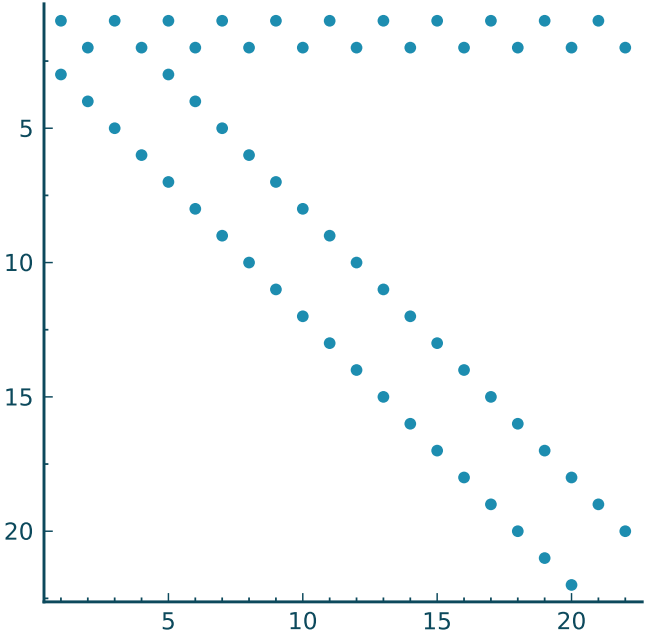
Ultraspherical methods

We have $\frac{d}{d\theta} T_k = kU_{k-1}$. Choose $\varphi_k = U_k$, but represent input in $\{T_k\}_{k=0}^N$.

See Jarlebring et al. (2010).

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Nonzeroes of E_N and A_N .

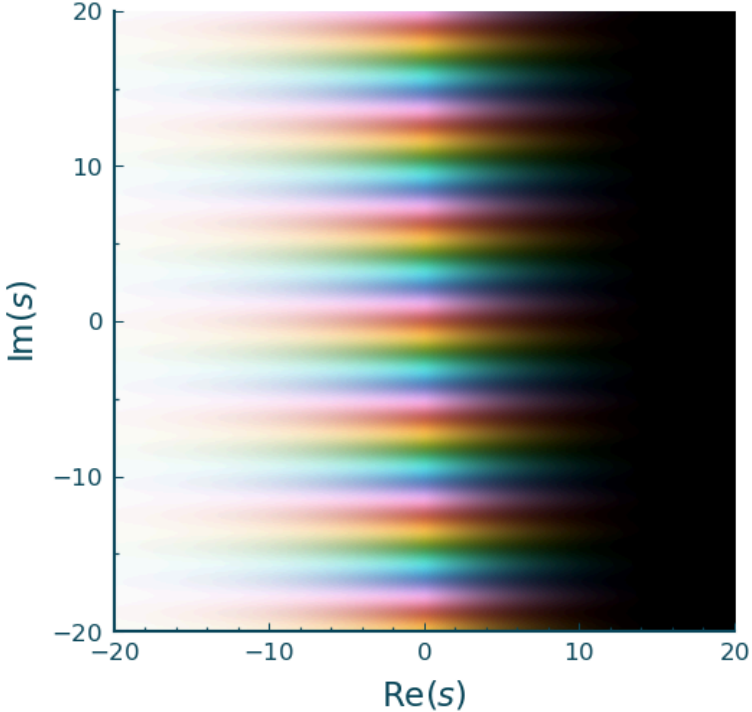
See Jarlebring et al. (2010).

Implicit rational approximation

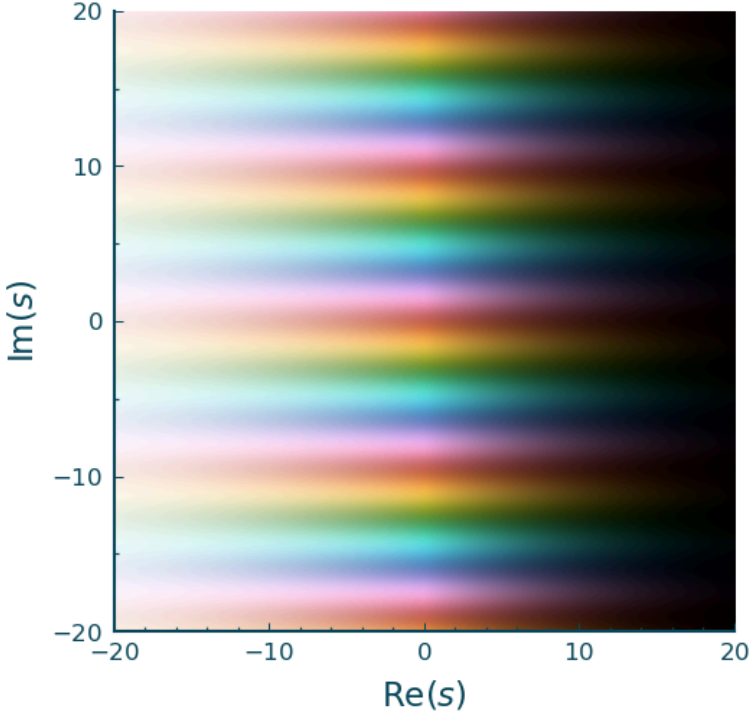
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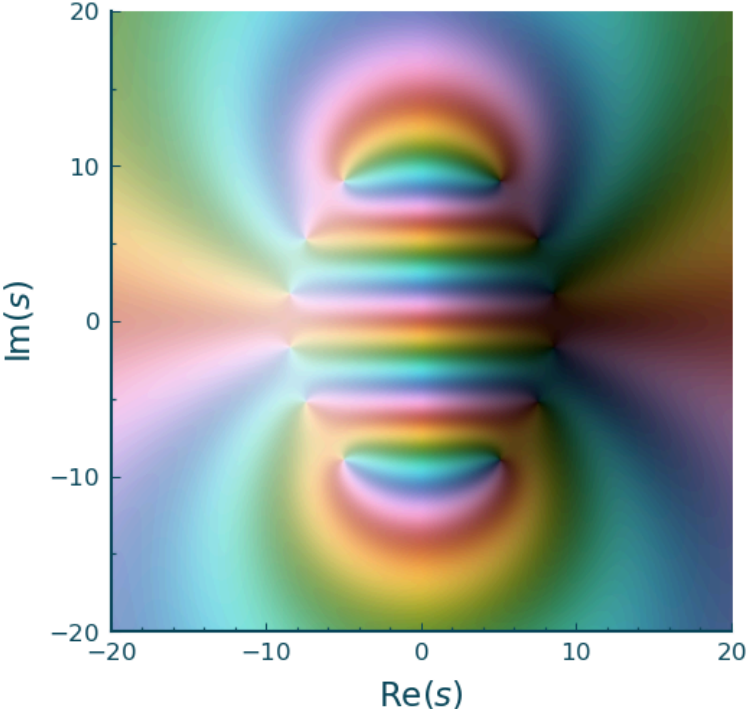
$\tau = \tau_m$



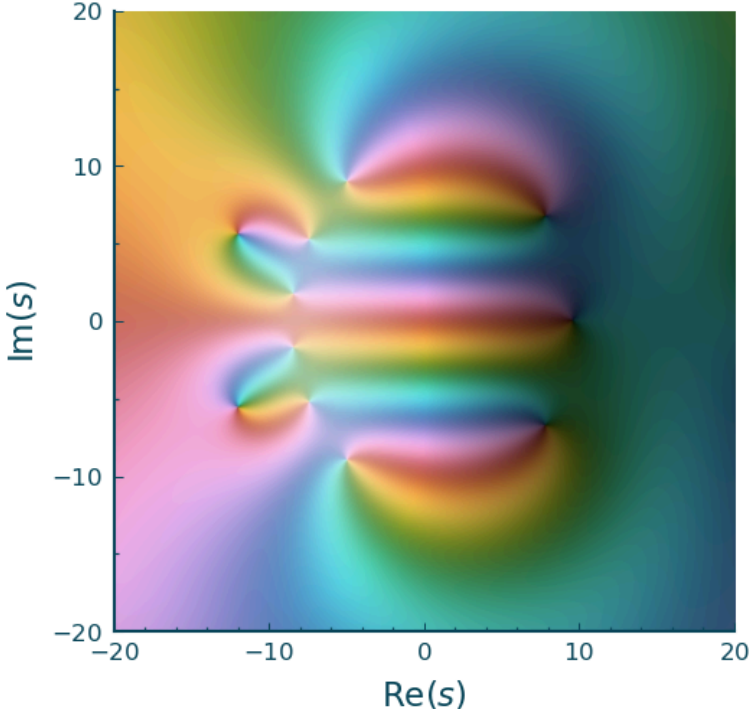
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Implicit rational approximation

$$G_N(s) = C(sI_n - \sum_{k=0}^m A_k r_N(s, -\tau_k))^{-1} B.$$



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See P. & Michiels (2024).

Implicit rational approximation

$$G_N(s) = C(sI_n - \sum_{k=0}^m A_k r_N(s, -\tau_k))^{-1} B,$$

where

$$\begin{cases} r_N(s, 0) = 1, \\ \mathcal{D}r_N(s, \cdot) = s\mathcal{T}_{N-1}r_N(s, \cdot). \end{cases}$$

See P. & Michiels (2024).

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$$r_N(s, \theta) = \frac{\sum_{k=0}^N \varphi_N^{(N-k)}(\theta) s^k}{\sum_{k=0}^N \varphi_N^{(N-k)}(0) s^k}.$$

See P. & Michiels (2024).

??? approximation for $m = 1$

Take $\varphi_N(\theta) = P_N^*(\theta)$, then

$$r_N(s, -\tau_m) = \dots = \frac{{}_1F_1[-N, -2N, -\tau_m s]}{{}_1F_1[-N, -2N, \tau_m s]}.$$

... = tedious steps, see P. & Michiels (2024).

Padé approximation for $m = 1$

Take $\varphi_N(\theta) = P_N^*(\theta)$, then

$$r_N(s, -\tau_m) = \dots = \frac{{}_1F_1[-N, -2N, -\tau_m s]}{{}_1F_1[-N, -2N, \tau_m s]}.$$

This is the N th diagonal Padé approximant of $e^{-\tau_m s}$!

... = tedious steps, see P. & Michiels (2024).

Preservation of stability for $m = 1$

1. Diagonal Padé approximants of $e^{-\tau s}$ satisfy

$$|r_N(s, -\tau)| \leq 1 \quad \text{for } s \in \mathbb{C}_{\geq 0}$$

See Ito (1985) and P. & Michiels (2026).

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Remember: $\det(sE_N - A_N) = \det(sI_n - A_0 - A_1 r_N(s, -\tau))$.

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Theorem

A Lanczos tau discretization of an exponentially stable single delay system using a Legendre basis is stable for N sufficiently large.

See Ito (1985) and P. & Michiels (2026).

The H^2 -norm

Approximating the H^2 -norm

For exponentially stable systems

$$\|G\|_{H^2} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \|G(i\omega)\|_F^2 d\omega \right)^{\frac{1}{2}}.$$

See Vanbiervliet et al. (2011).

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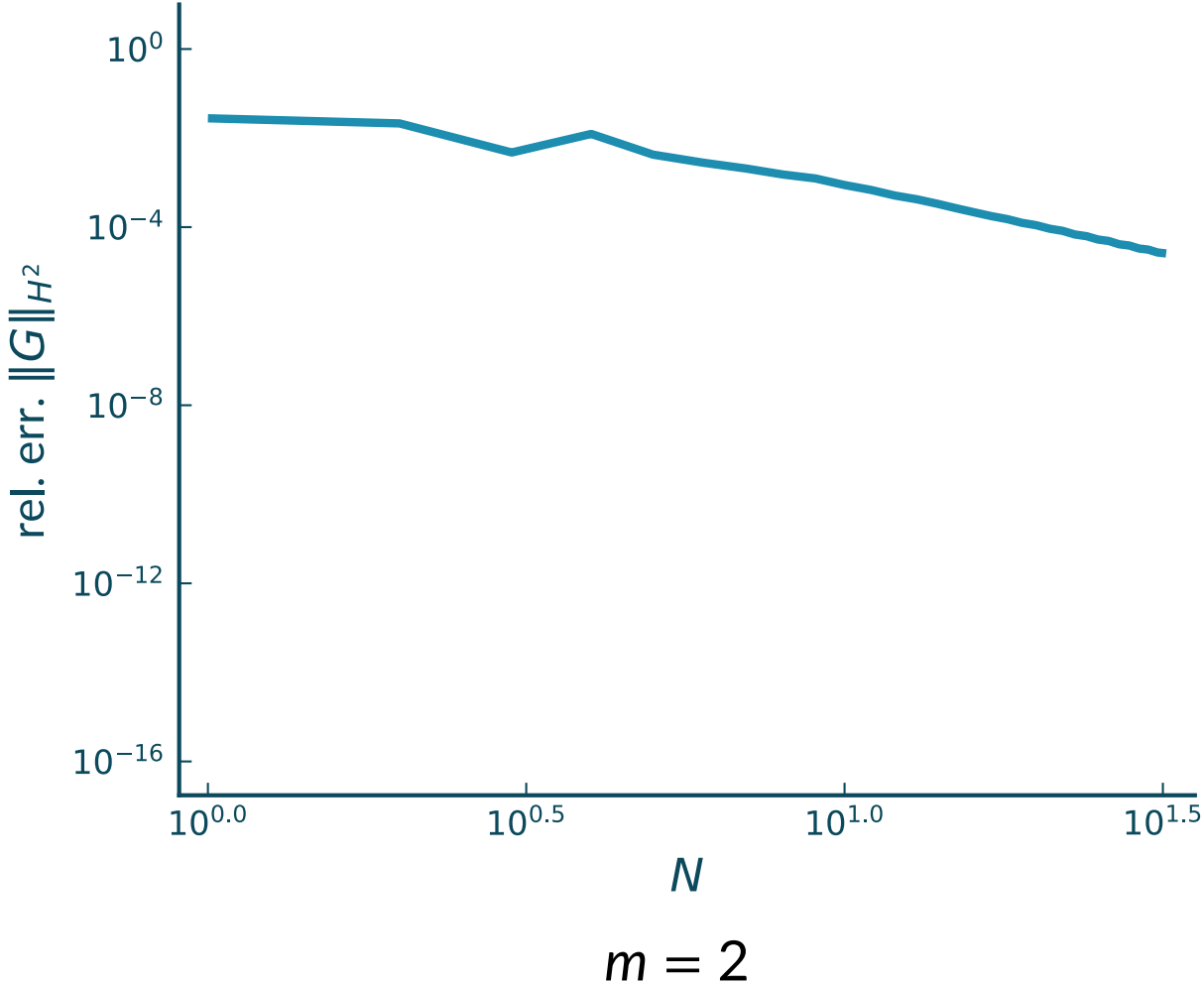
$$\|G\|_{H^2} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \|G(i\omega)\|_F^2 d\omega \right)^{\frac{1}{2}}.$$

Idea: $\|G\|_{H^2} \approx \|G_N\|_{H^2} = \sqrt{\text{tr}(C_N V C_N^T)}$, where

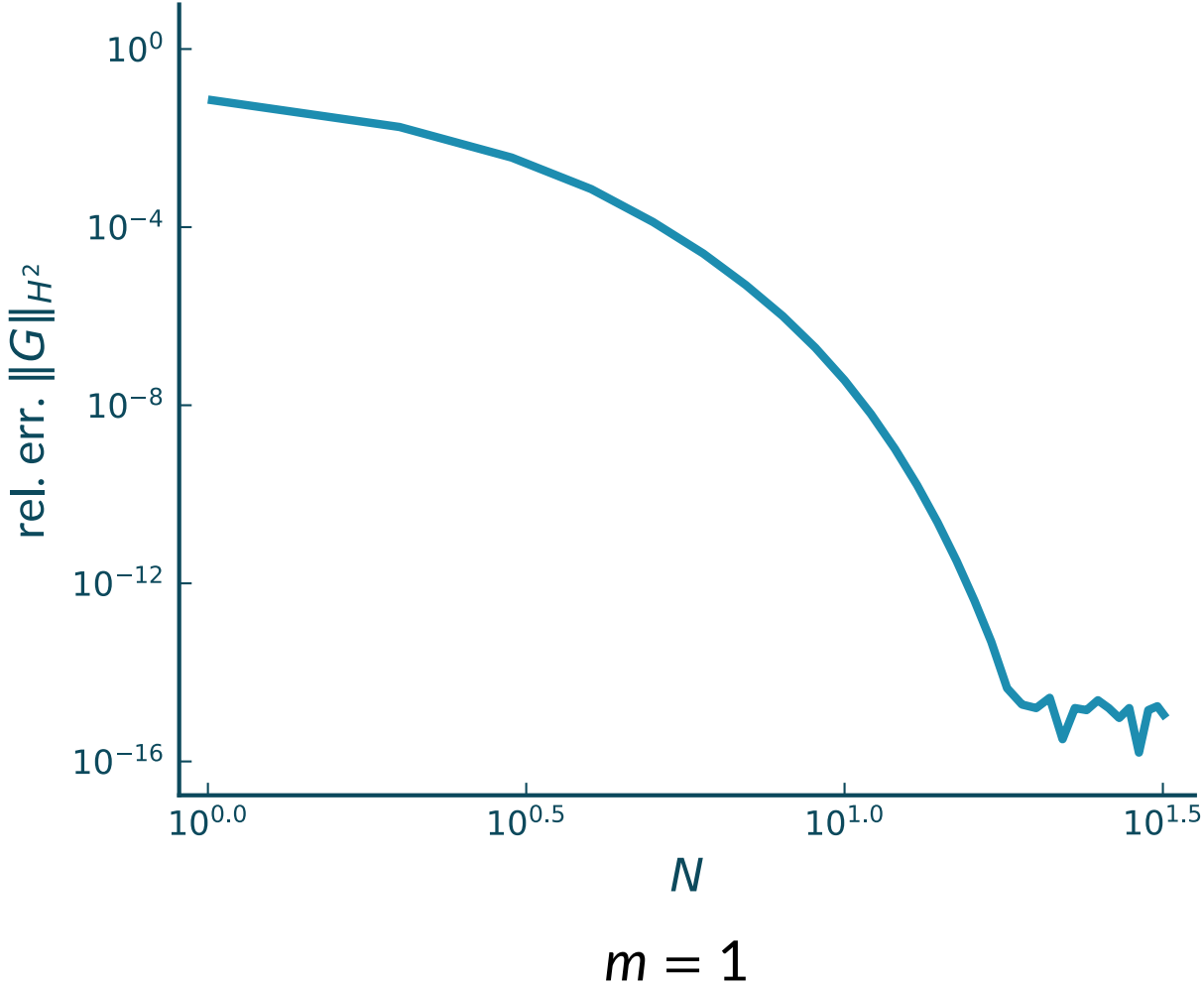
$$A_N V E_N^T + E_N V A_N^T = -B_N B_N^T.$$

See Vanbiervliet et al. (2011).

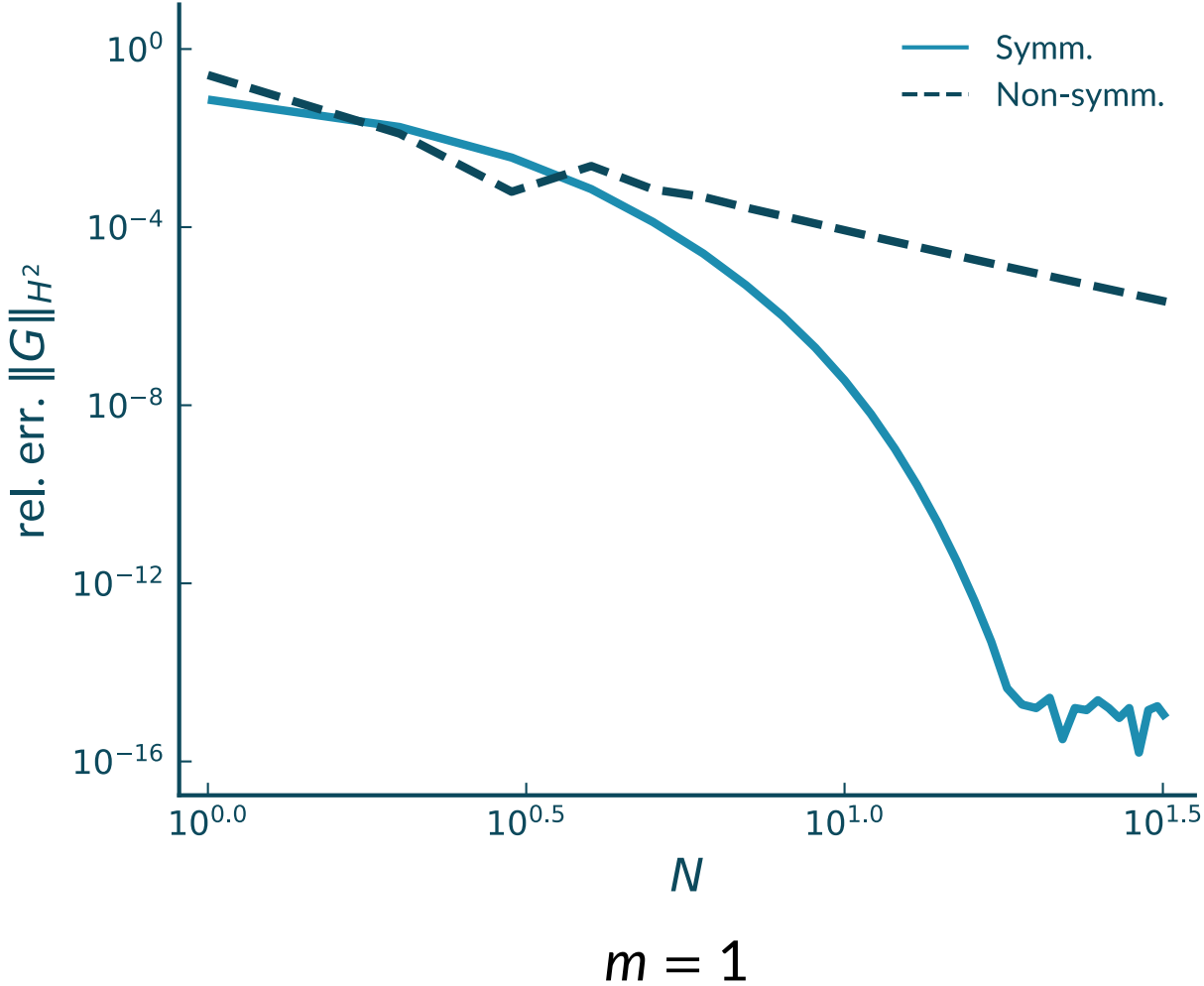
Convergence rates



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A hint

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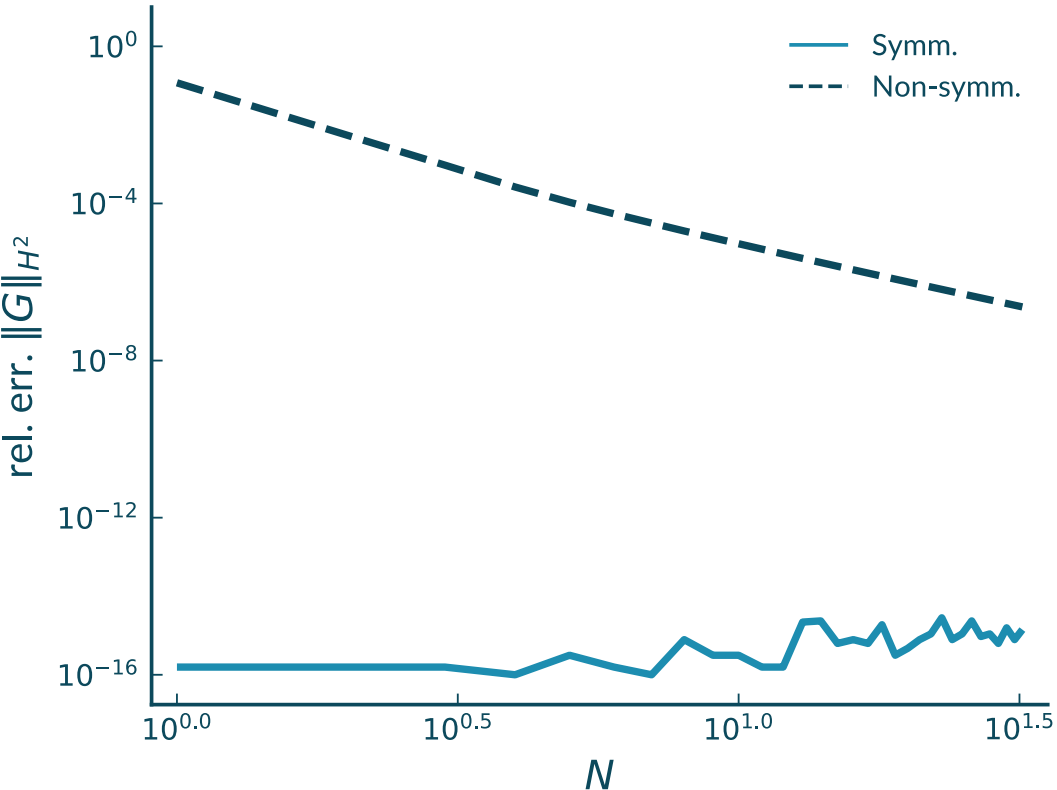
Super convergence

Let $m = 1$ and $A_0 = A_1 = a < 0$.

See P. & Michiels (2024).

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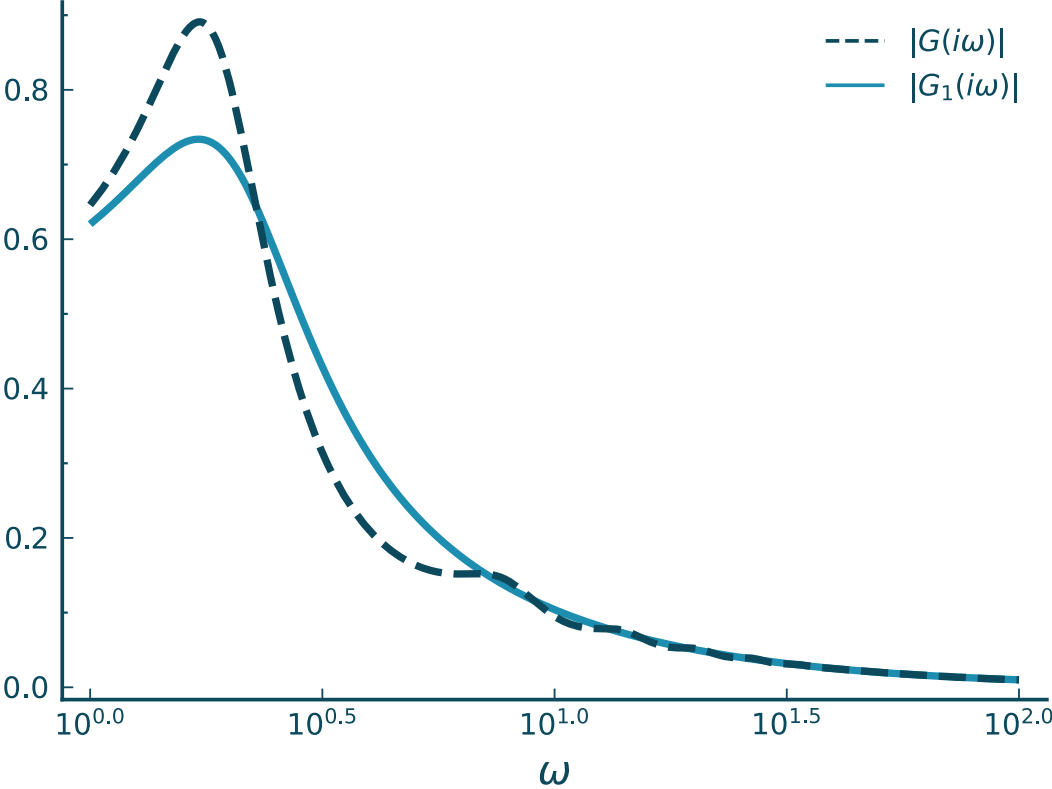
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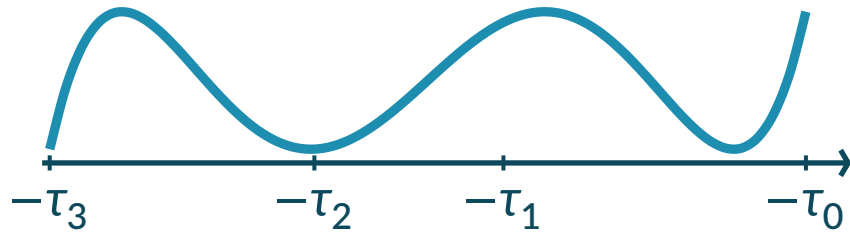
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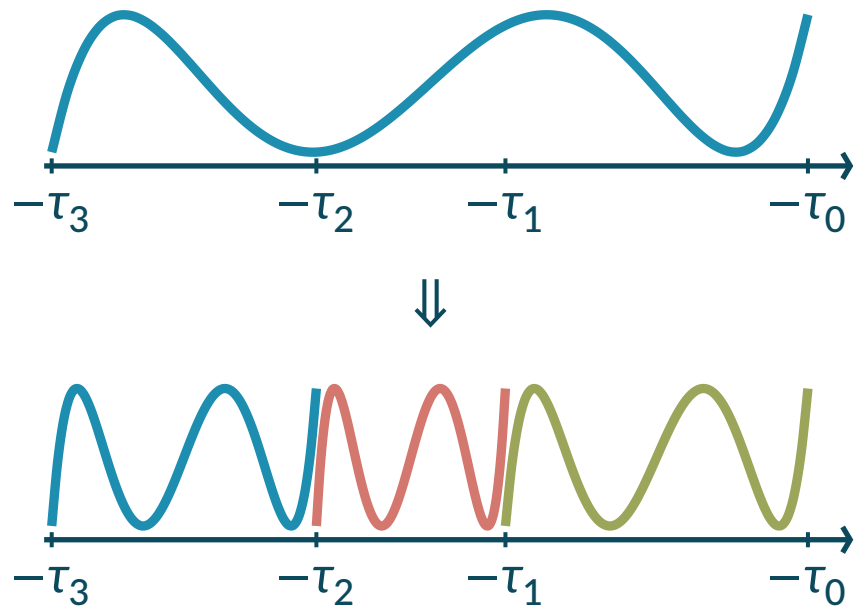
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Extension: Accelerating convergence using splines



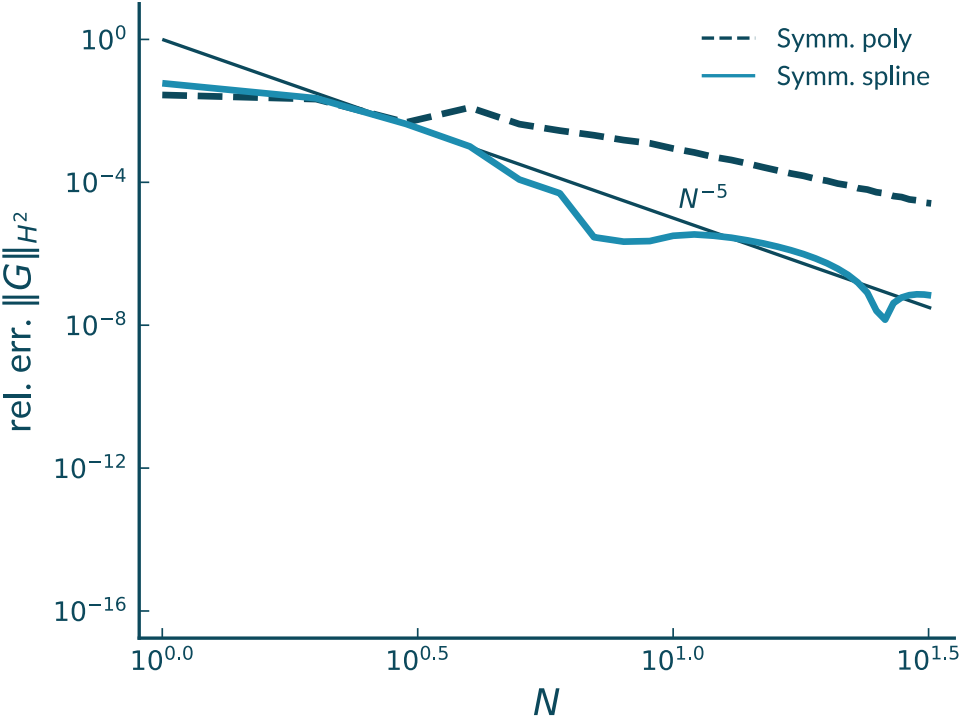
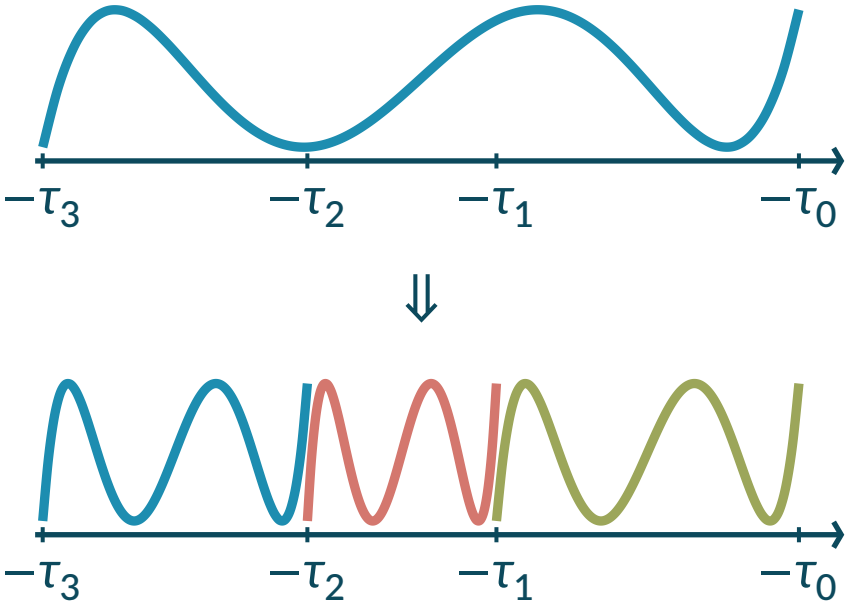
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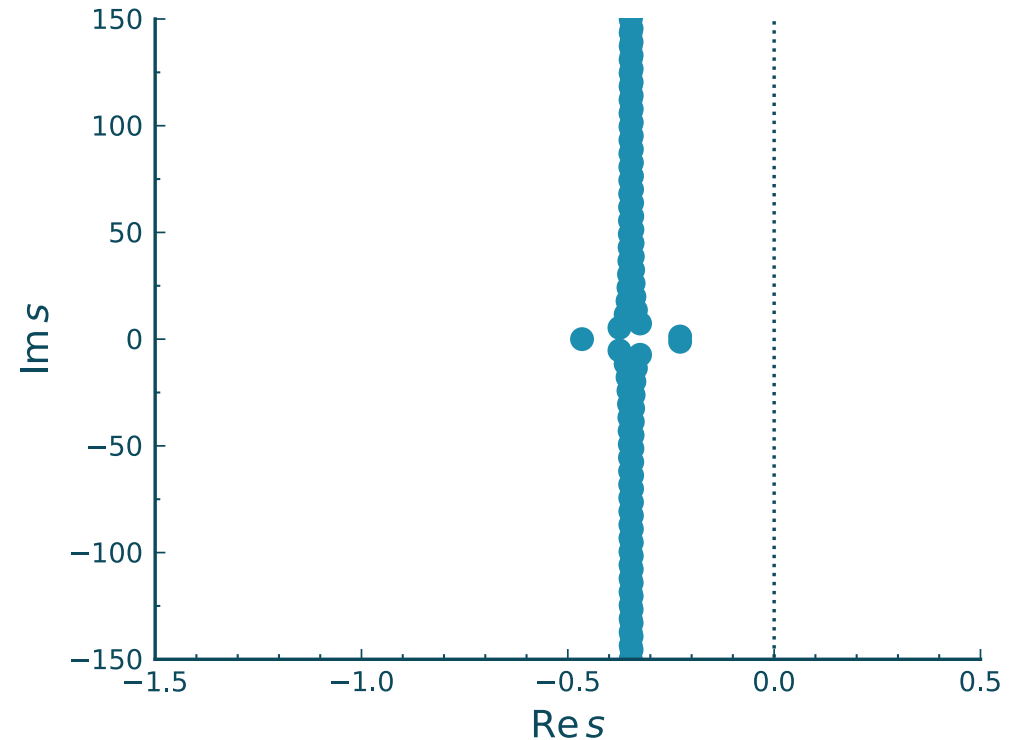
Extension: Delay differential algebraic equations

$$E\dot{\mathbf{x}}(t) = \sum_{k=0}^m A_k \mathbf{x}(t - \tau_k) + B\mathbf{u}(t),$$
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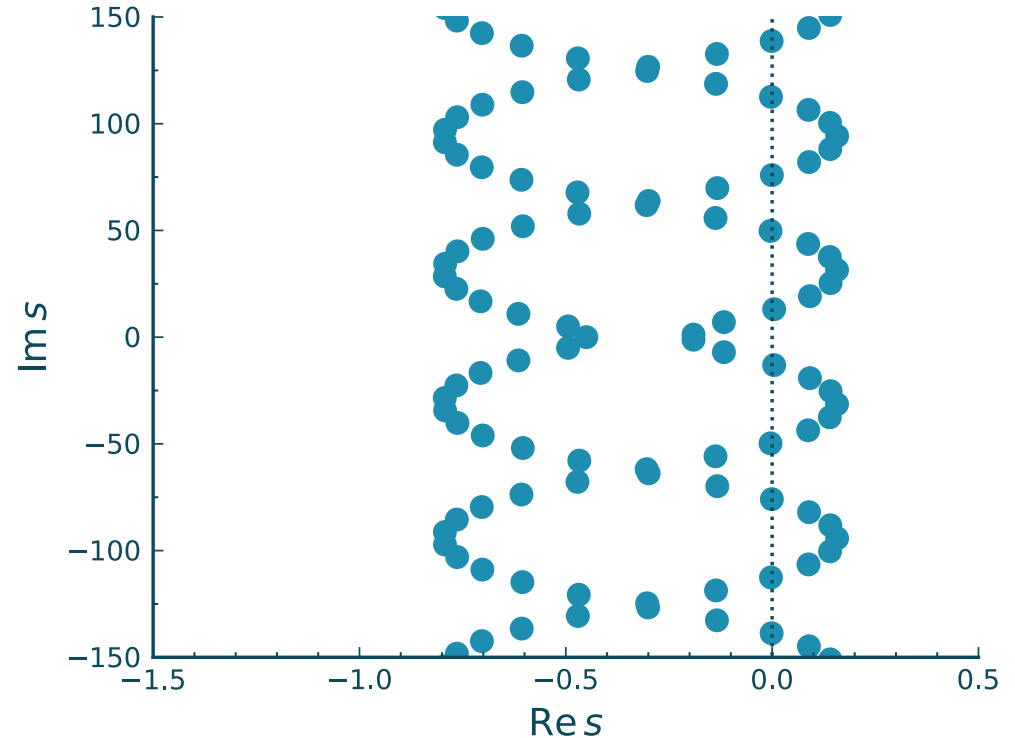
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Conclusions

Discretize-first spectral methods work great for global characteristics of linear time-delay systems: e.g. stability analysis and the H^2 -norm.

Symmetry improves convergence for the H^2 -norm of single delay systems.*

For theoretical guarantees: use a Legendre basis due to its connection to Padé approximation.

For large scale problems: use Chebyshev polynomials of the second kind (i.e. infinite Arnoldi).

* Not in this talk: this can be extended to multiple discrete delays through the use of splines.