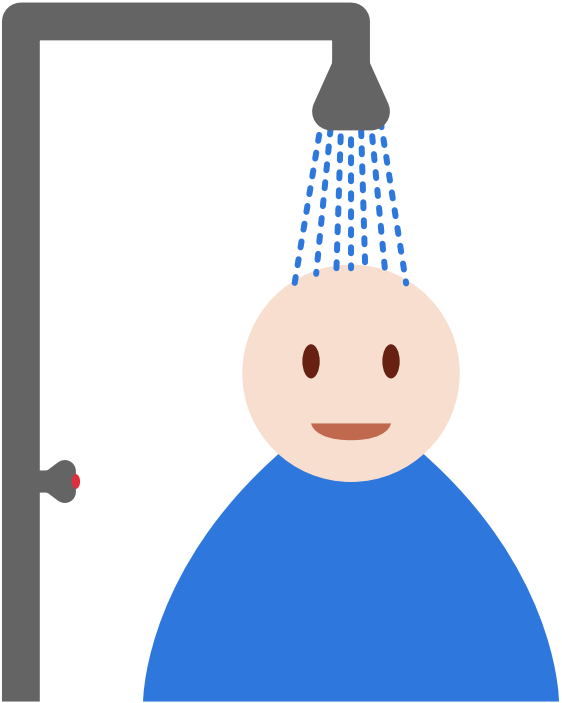


Lanczos Tau Methods for Time-Delay Systems

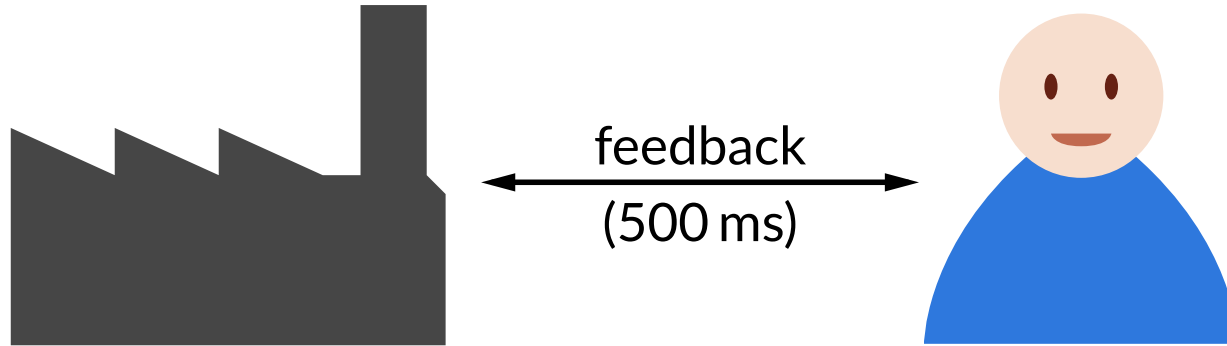
Approximation of System and its H^2 -norm

Evert Provoost

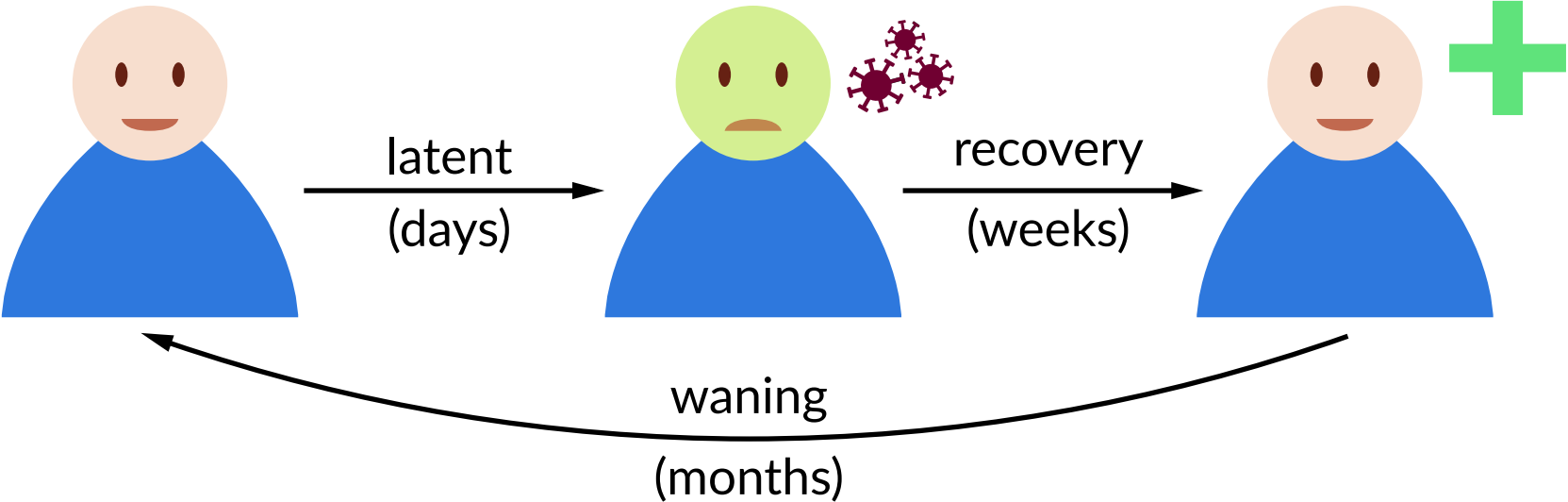
Delay is everywhere



Delay is everywhere

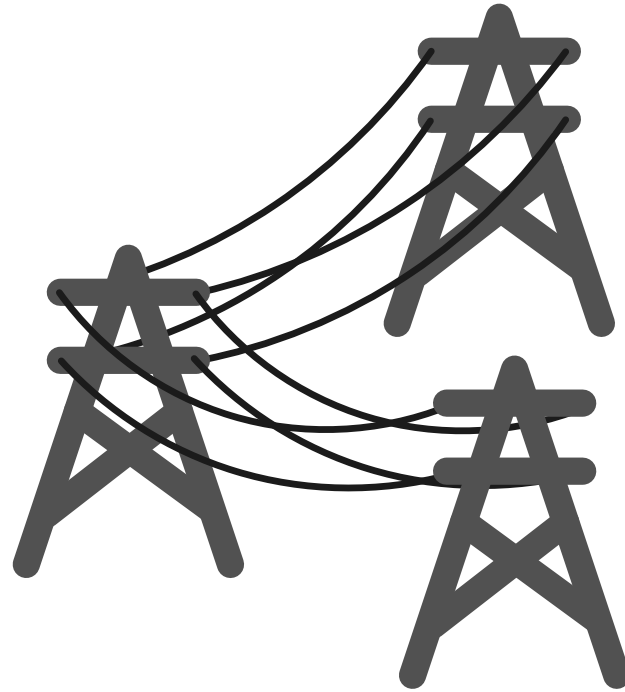


Delay is everywhere

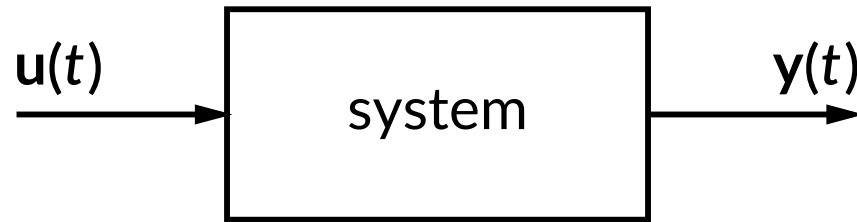


Delay is everywhere

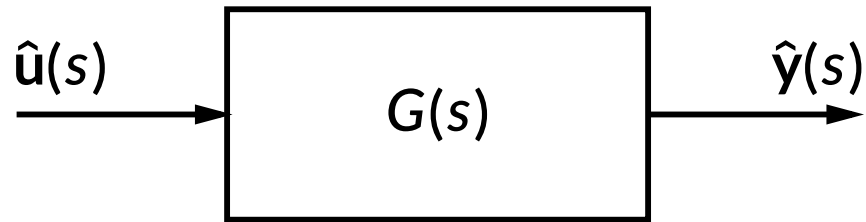
transmission delays



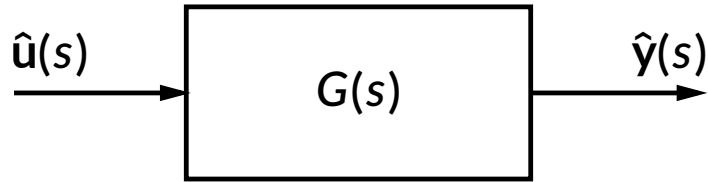
What is a system?



What is a system?

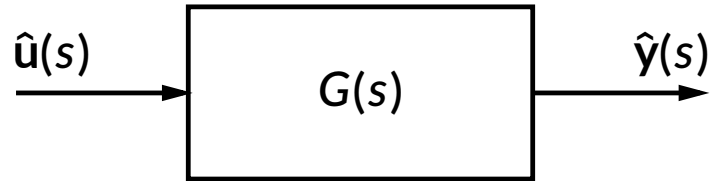


Why H^2 -norm?



$\|G\|_{H^2} \propto \text{amplification of noise}$

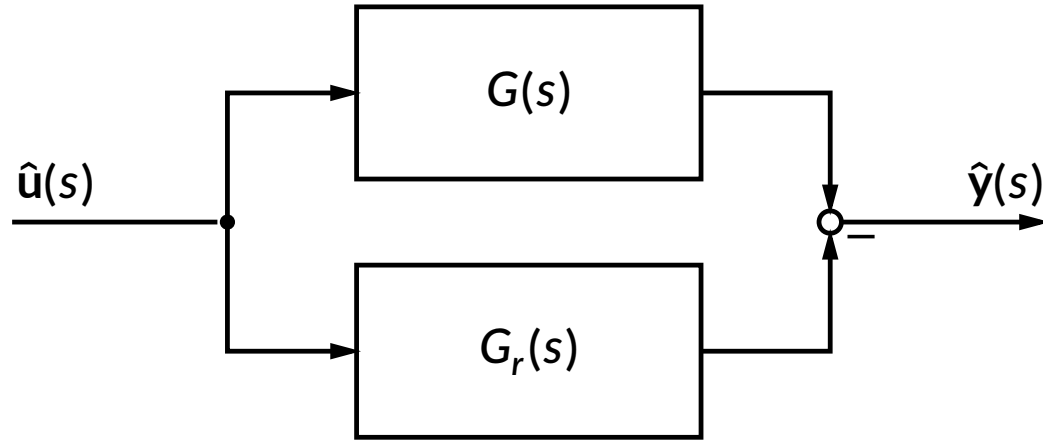
Why H^2 -norm?



$\|G\|_{H^2} \propto$ amplification of noise

$\|G\|_{H^2} < \infty \implies$ system is stable

Why H^2 -norm?



$$\|G - G_r\|_{H^2} \propto \text{approximation error}$$

The goal

Approximate $\|G\|_{H^2}$.

The plan

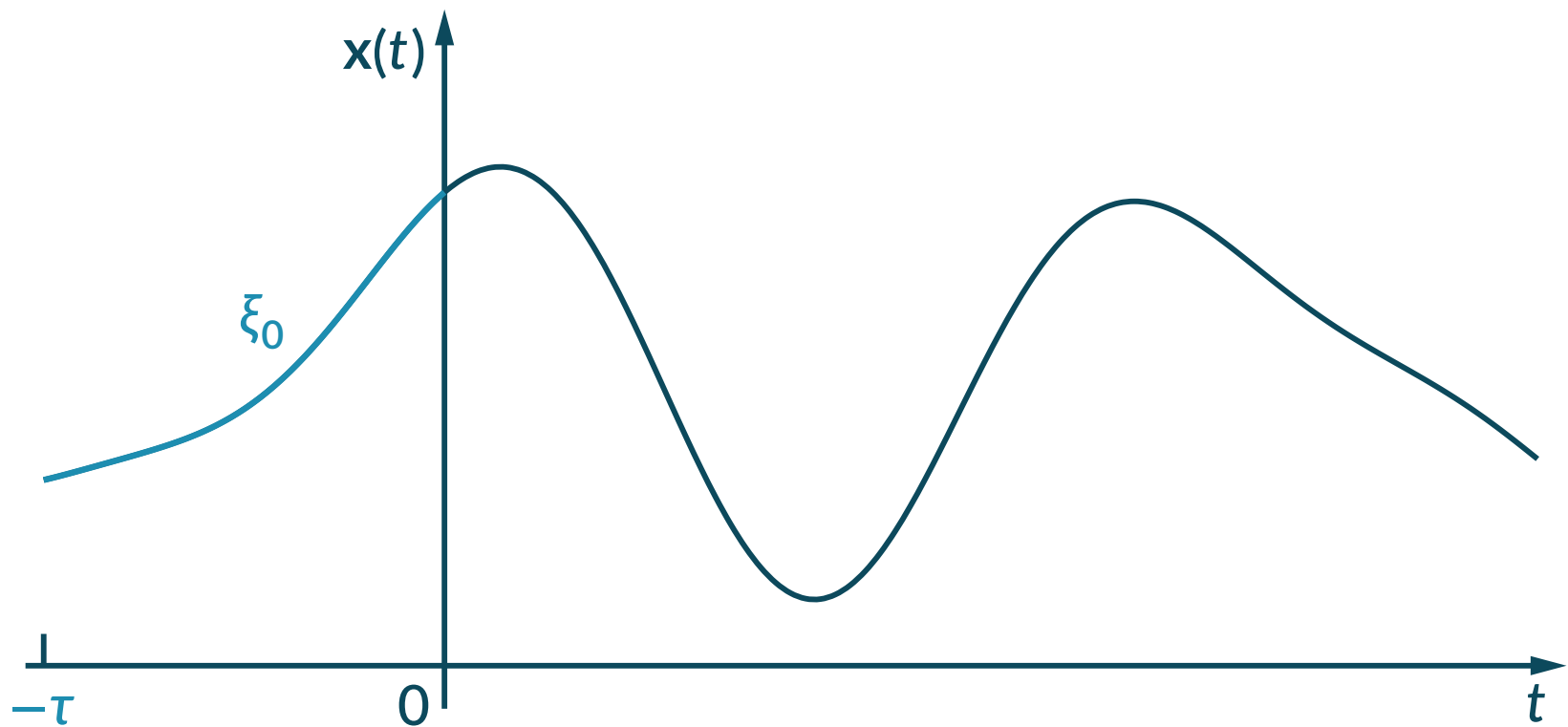
We know how to compute H^2 -norm for delay-free system.

The plan

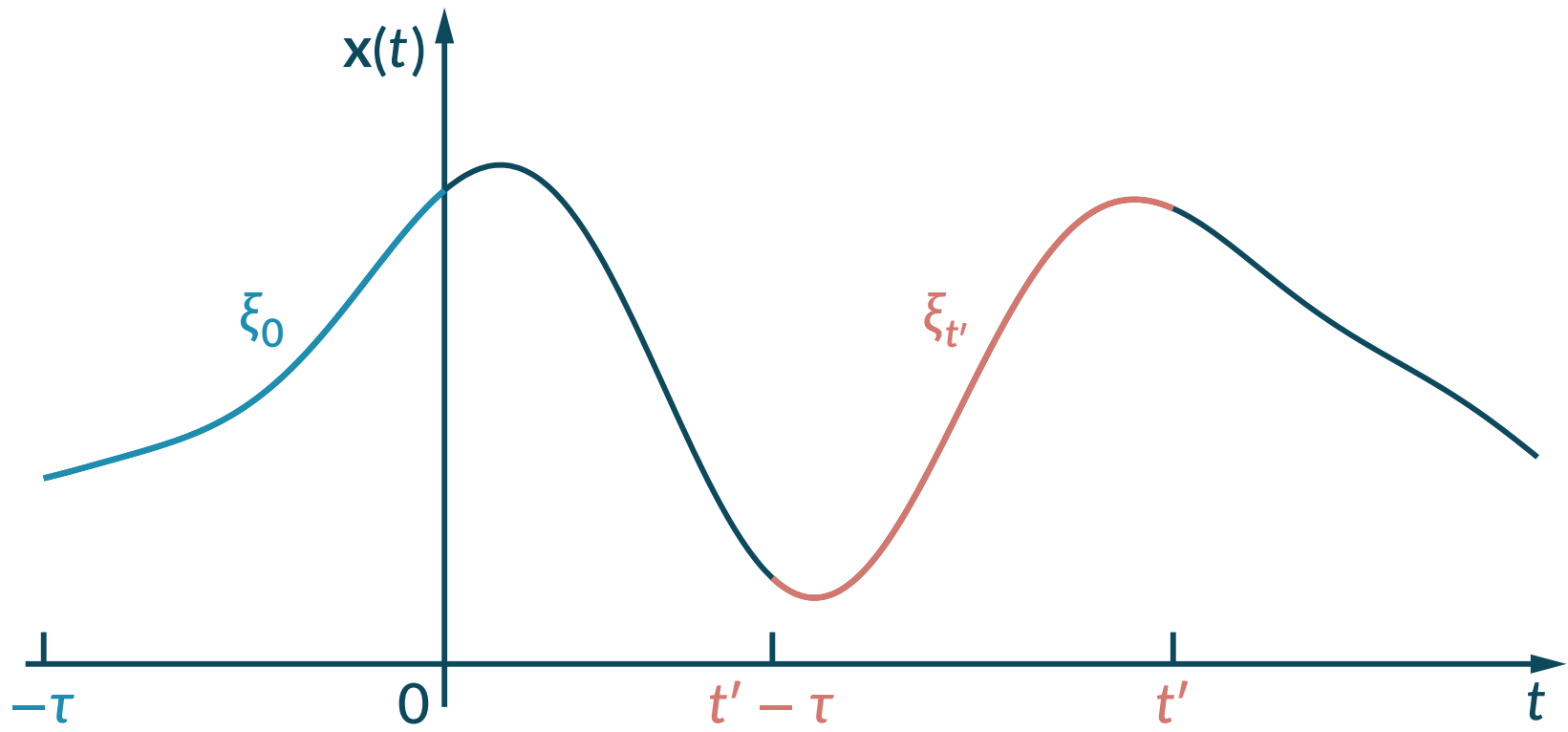
We know how to compute H^2 -norm for delay-free system.

1. Construct delay-free approximation G_N .
2. Use $\|G\|_{H^2} \approx \|G_N\|_{H^2}$.

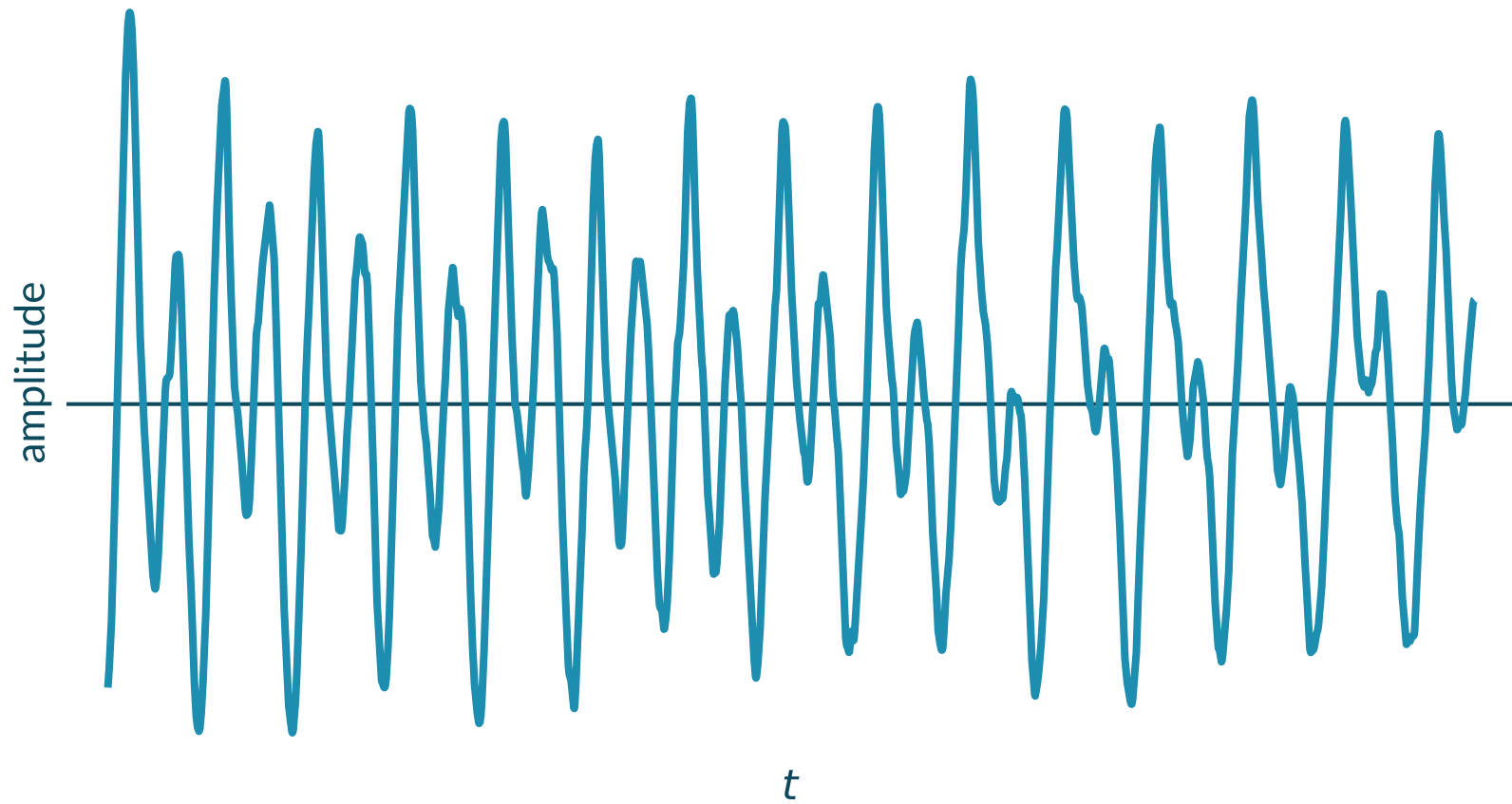
The history function



The history function

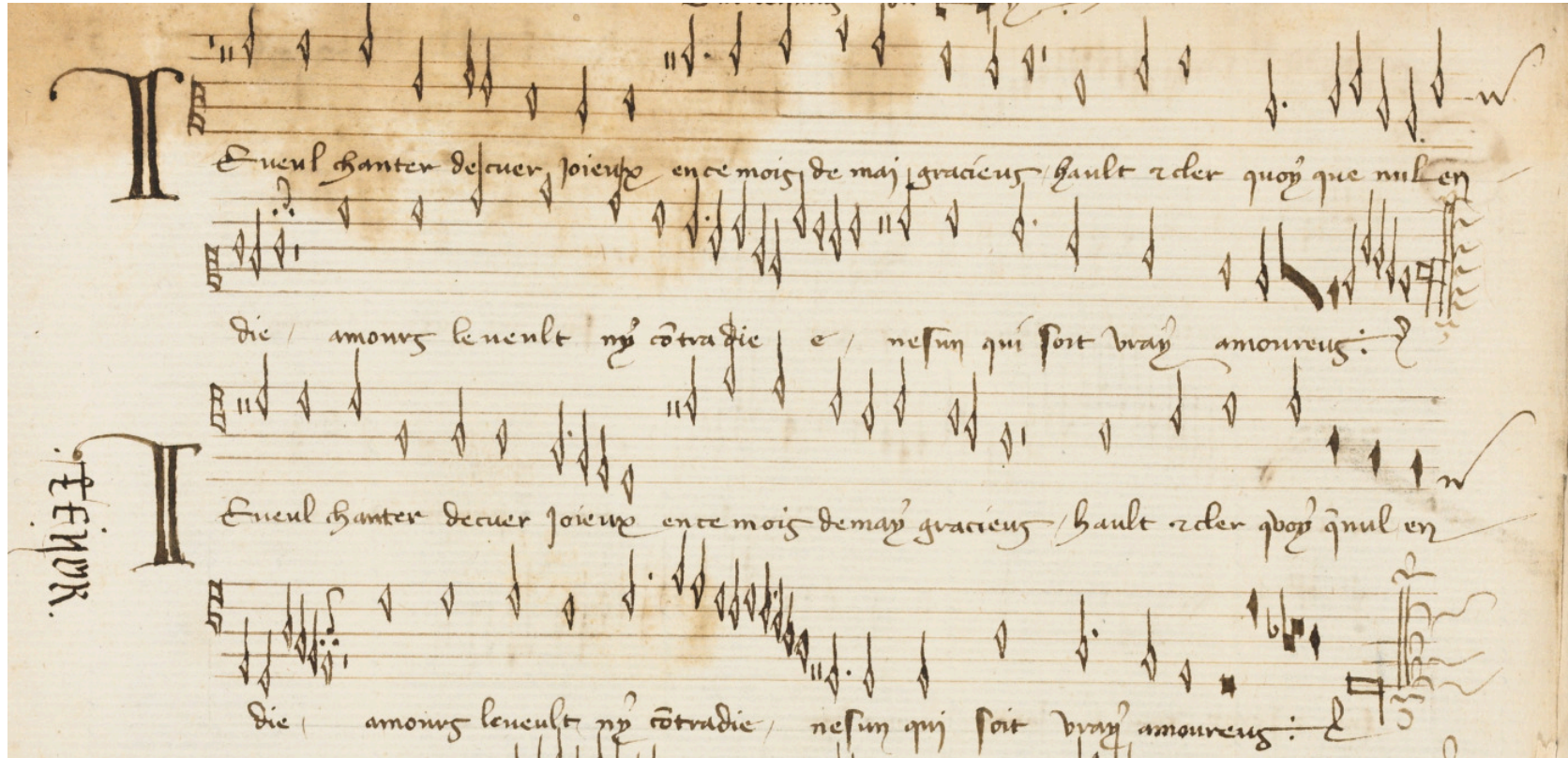


A musical detour



Fragment of G. Du Fay's 'Je veulx chanter' as performed by the Sollazzo Ensemble.

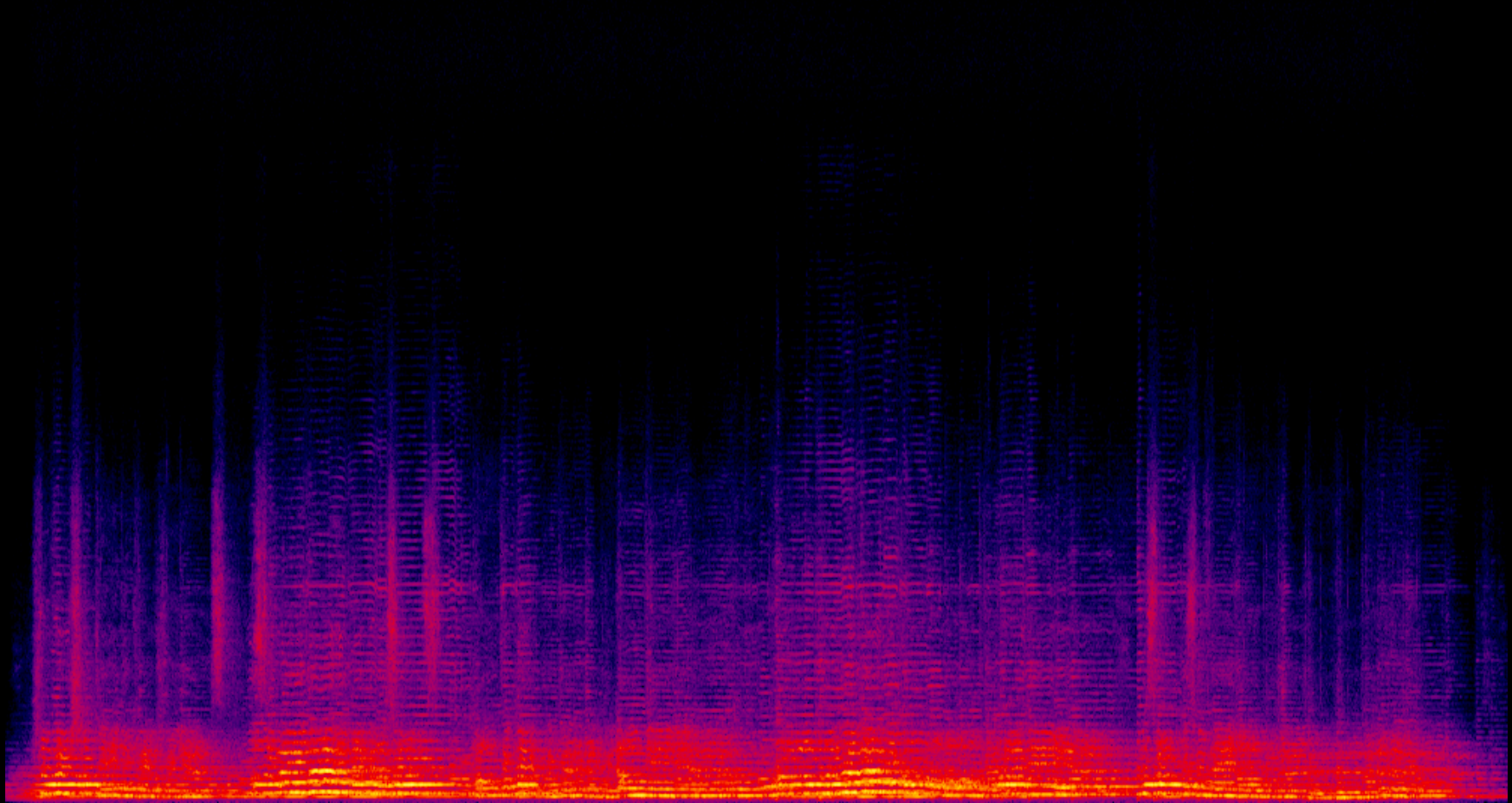
Real life functions have structure



Scan courtesy of the Bodleian Library, Oxford.

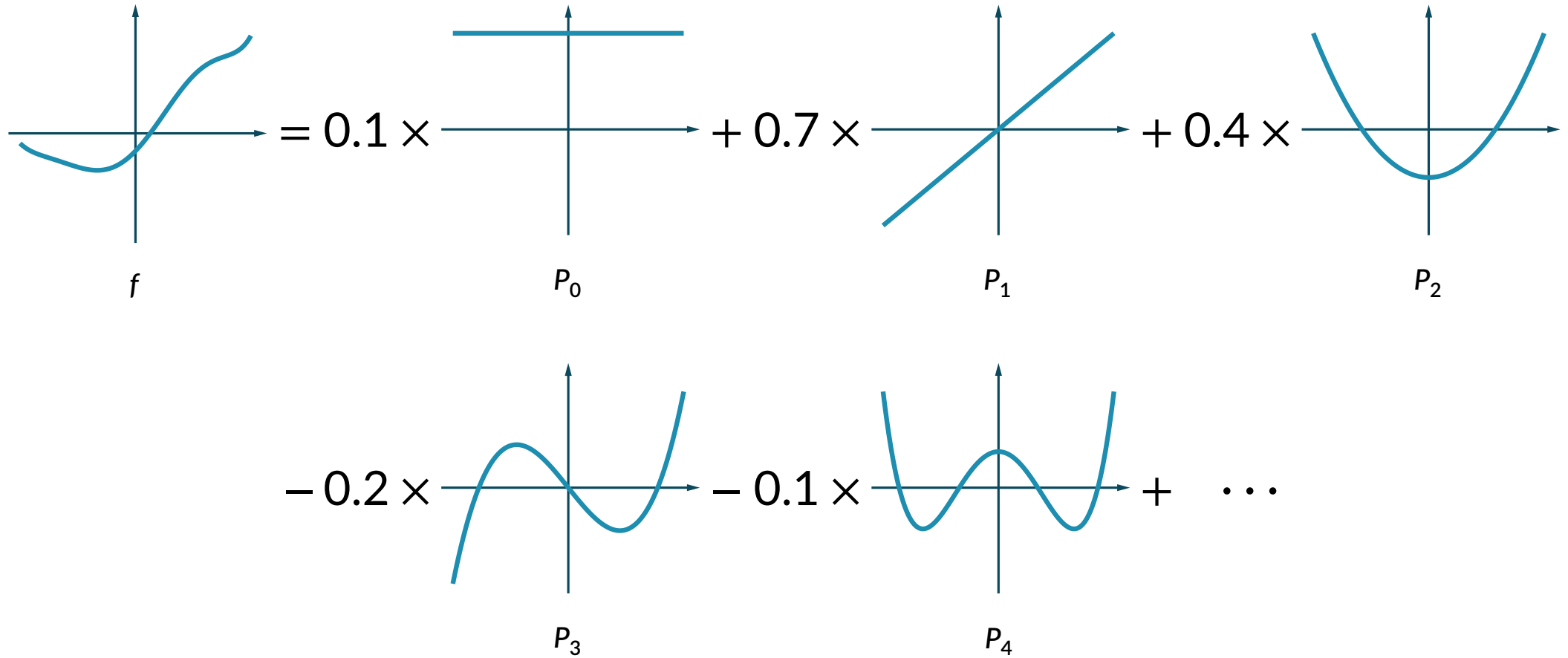
Real life functions have structure

frequency

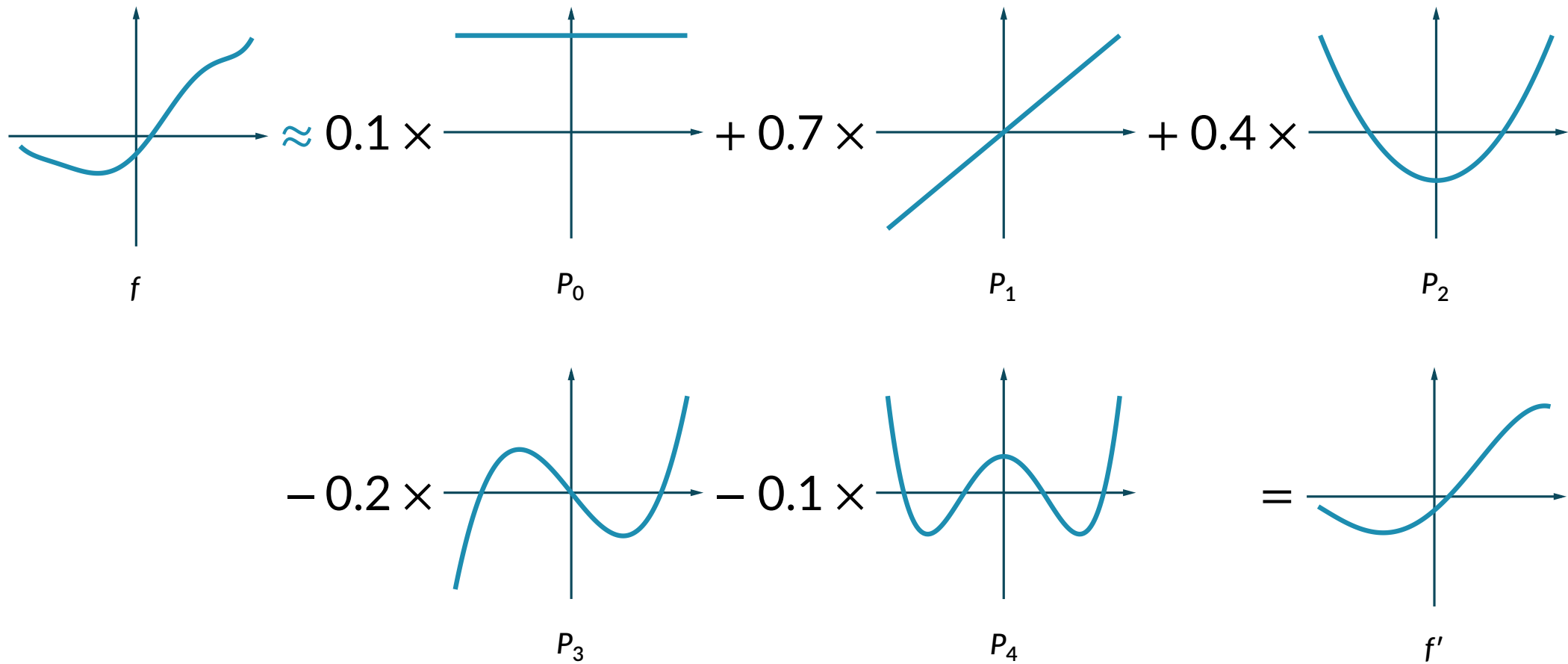


t

Representing functions as series of polynomials



Approximating functions as series of polynomials



The story thus far

$$G \implies G_N$$

$$\|G\|_{H^2} \approx \|G_N\|_{H^2}$$

The story thus far

$$G \implies G_N$$

Ito and Teglas (1986, *SIAM J. Sci. Comput.*, 24/4: 737–59)

$$\|G\|_{H^2} \approx \|G_N\|_{H^2}$$

Vanbiervliet, Michiels, and Jarlebring (2011, *Int. J. Control*, 84/2: 228–41)

What did I do?

- Extend to include algebraic constraints.
- Predict convergence rates.
- Improve convergence rates using splines.
- Prove that $\|G_N\|_{H^2} \rightarrow \|G\|_{H^2}$ as $N \rightarrow \infty$.
- Use this approximation method to minimize $\|G\|_{H^2}$.

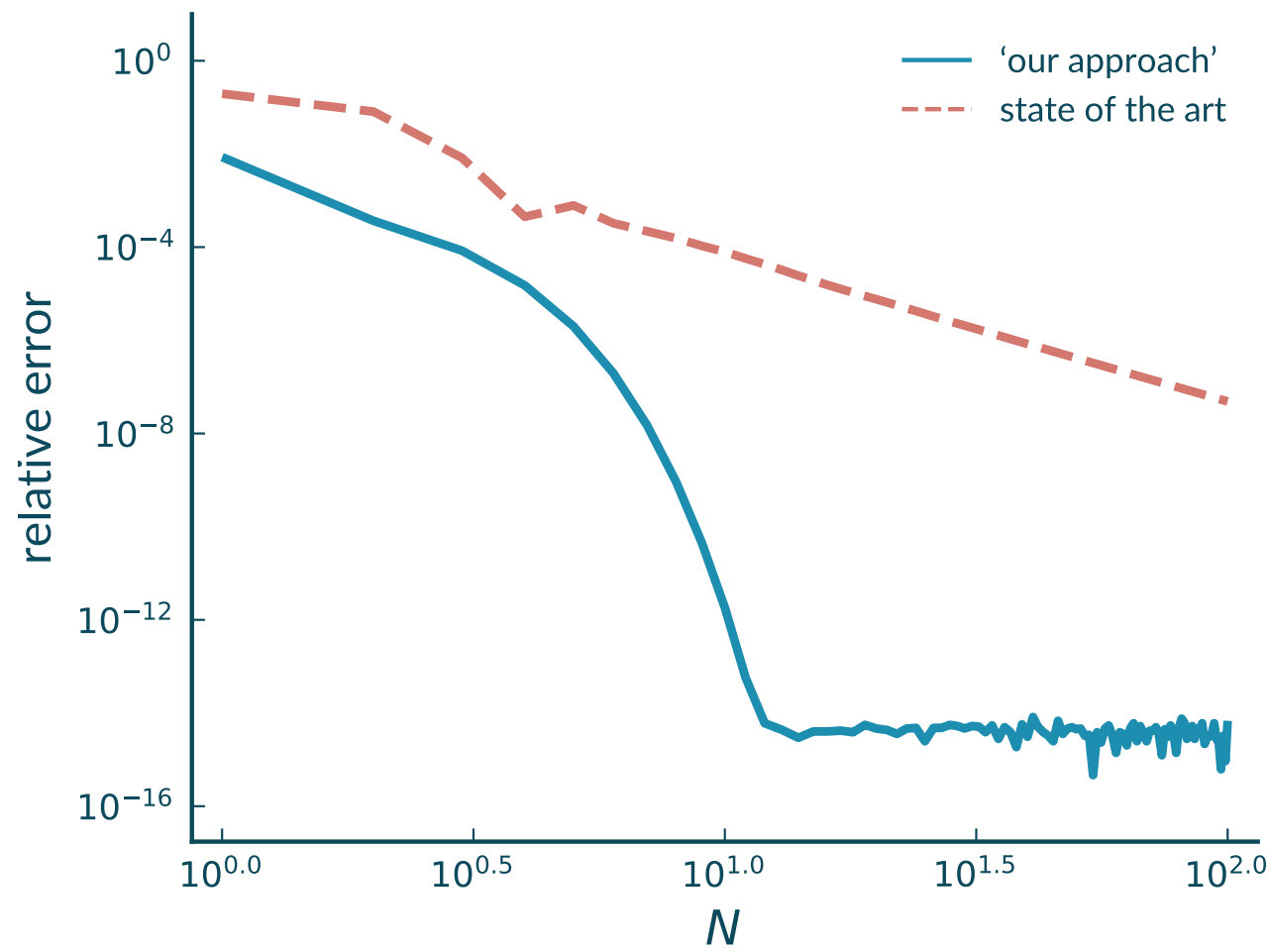
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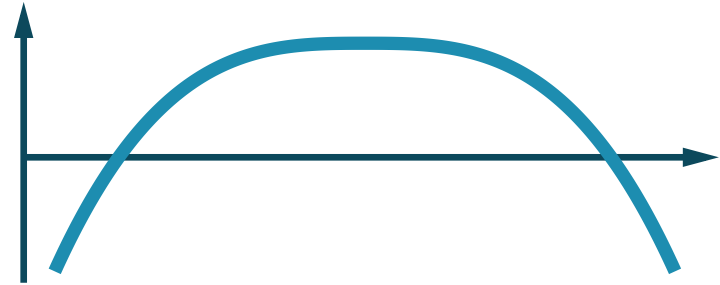
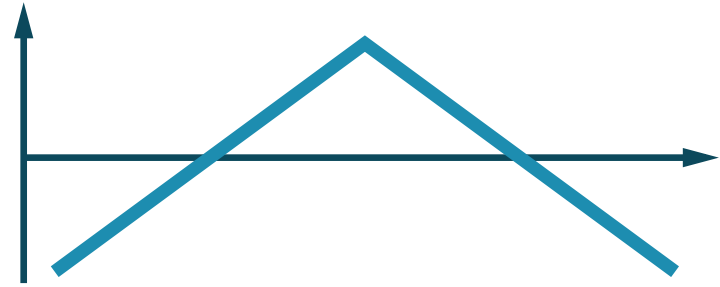
This talk: the highlights.

Convergence rate

Convergence is better than expected for single delay



What is smoothness?



Convergence can be understood in terms of a spectral method

1. Polynomials approximate better the smoother the function.

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2. Underlying the method is the delay-Lyapunov matrix $\lambda(\theta)$.

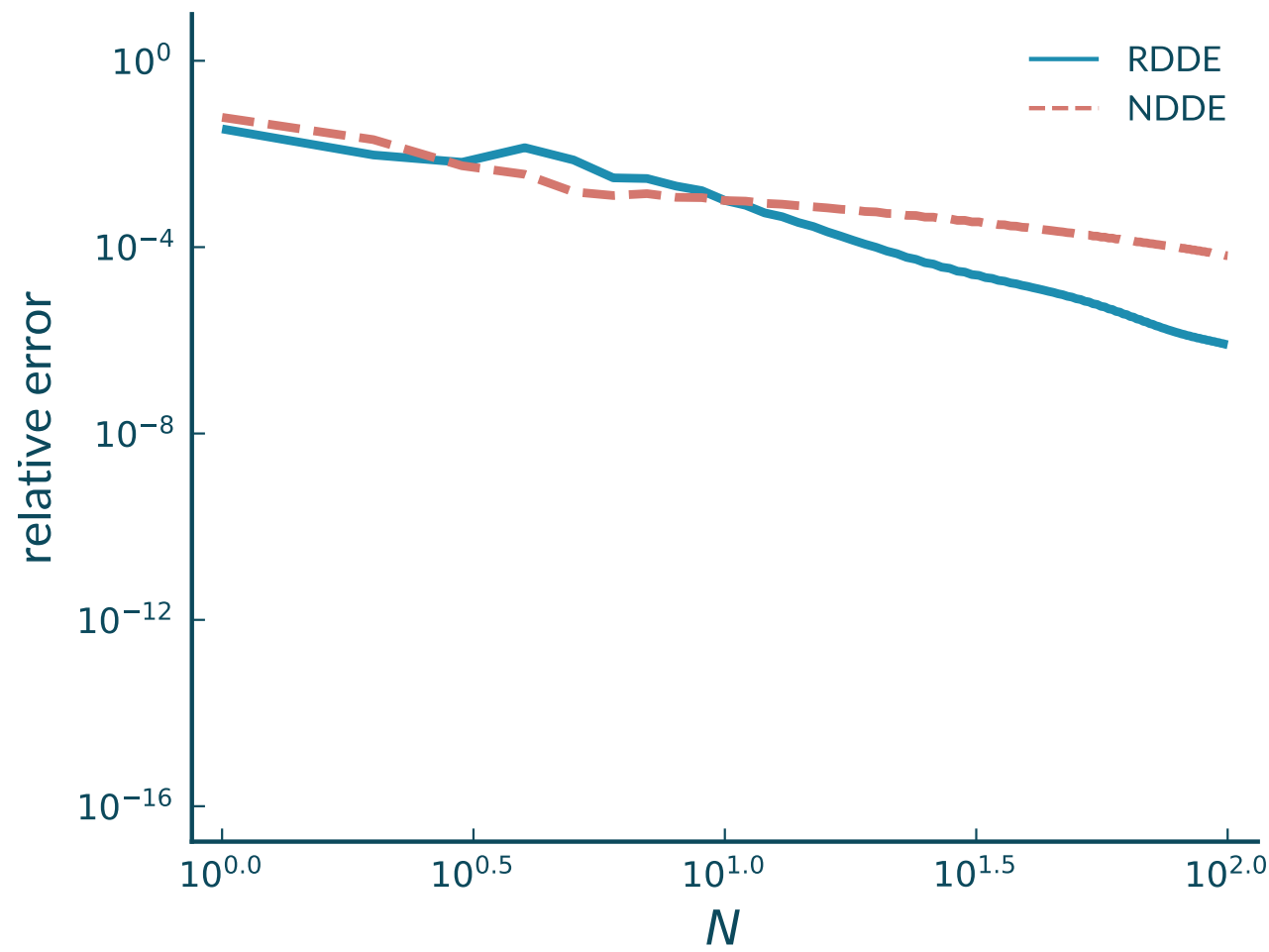
Convergence can be understood in terms of a spectral method

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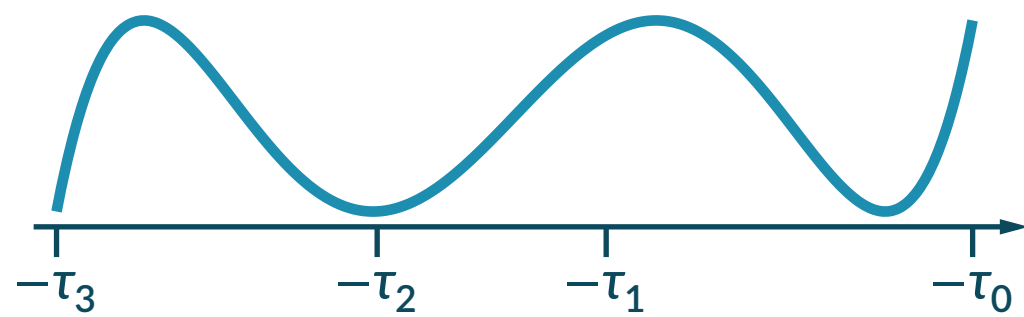
$\implies$ smoothness of $\lambda(\theta) \sim$ speed of convergence.

There are exceptions and symmetry is needed (neither of which I fully understand...)

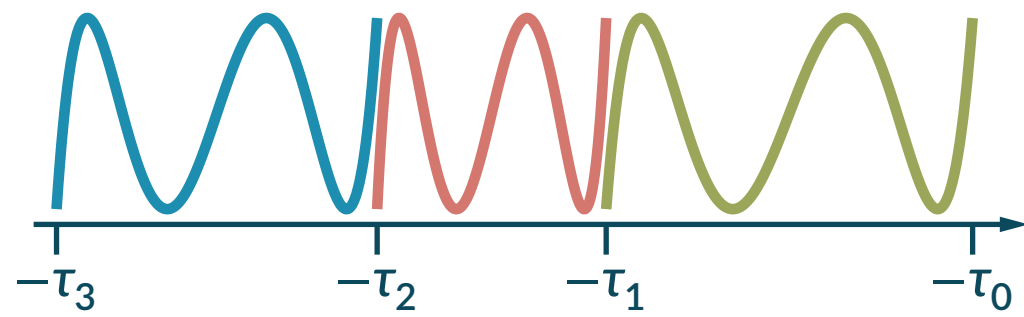
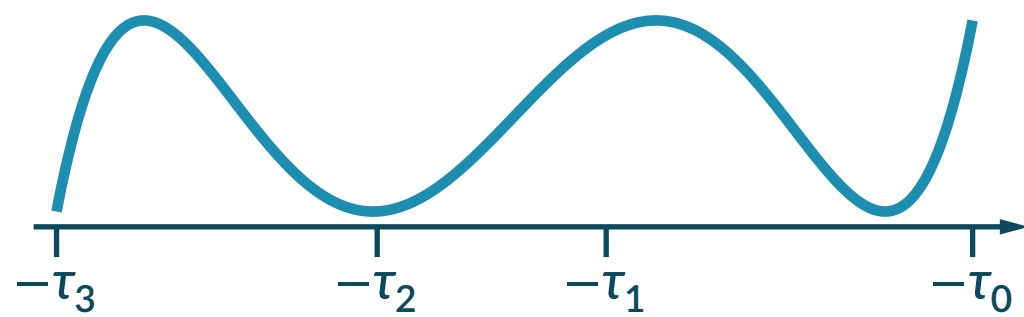
Reduces to algebraic for multiple delays



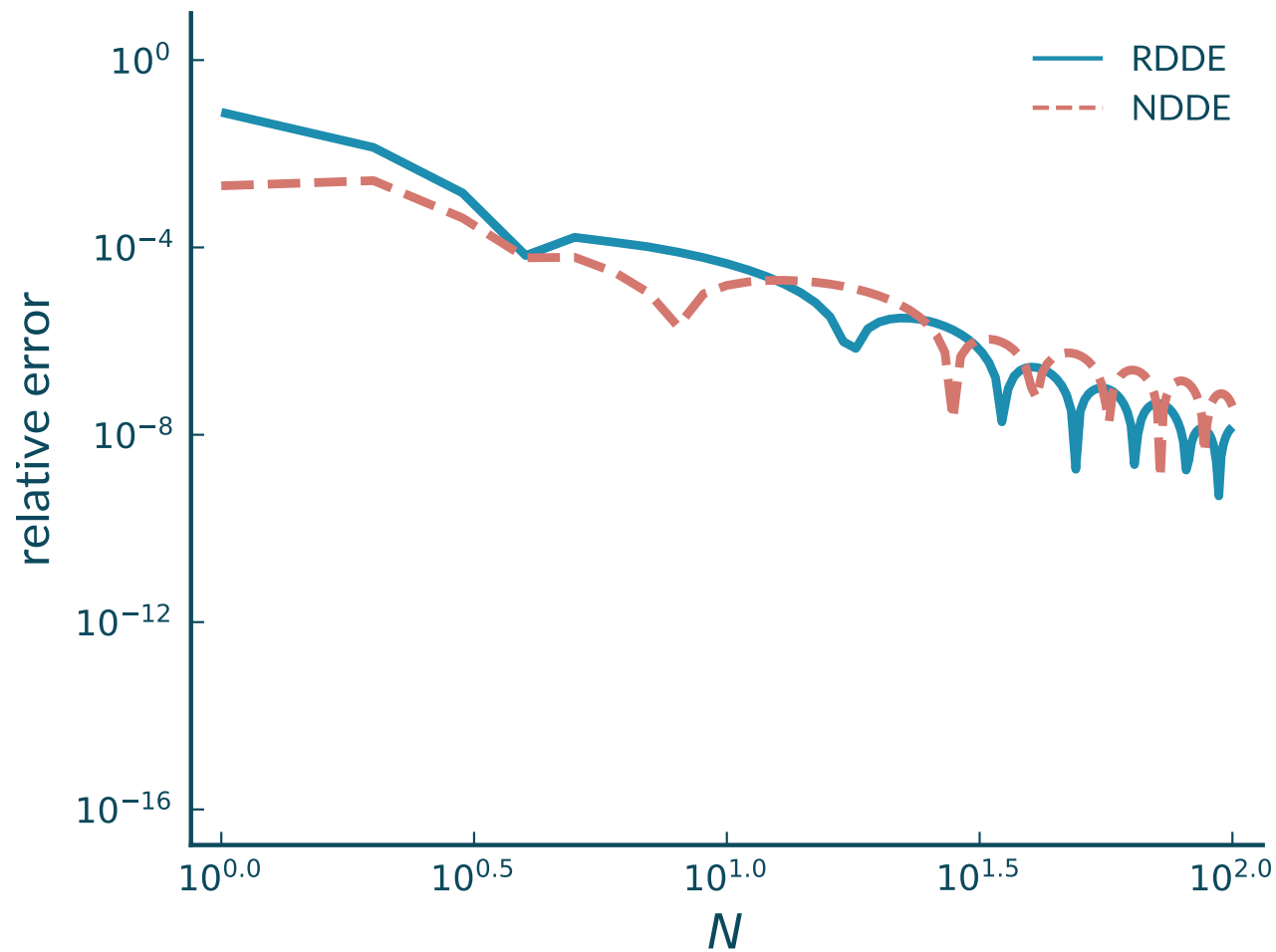
Splines



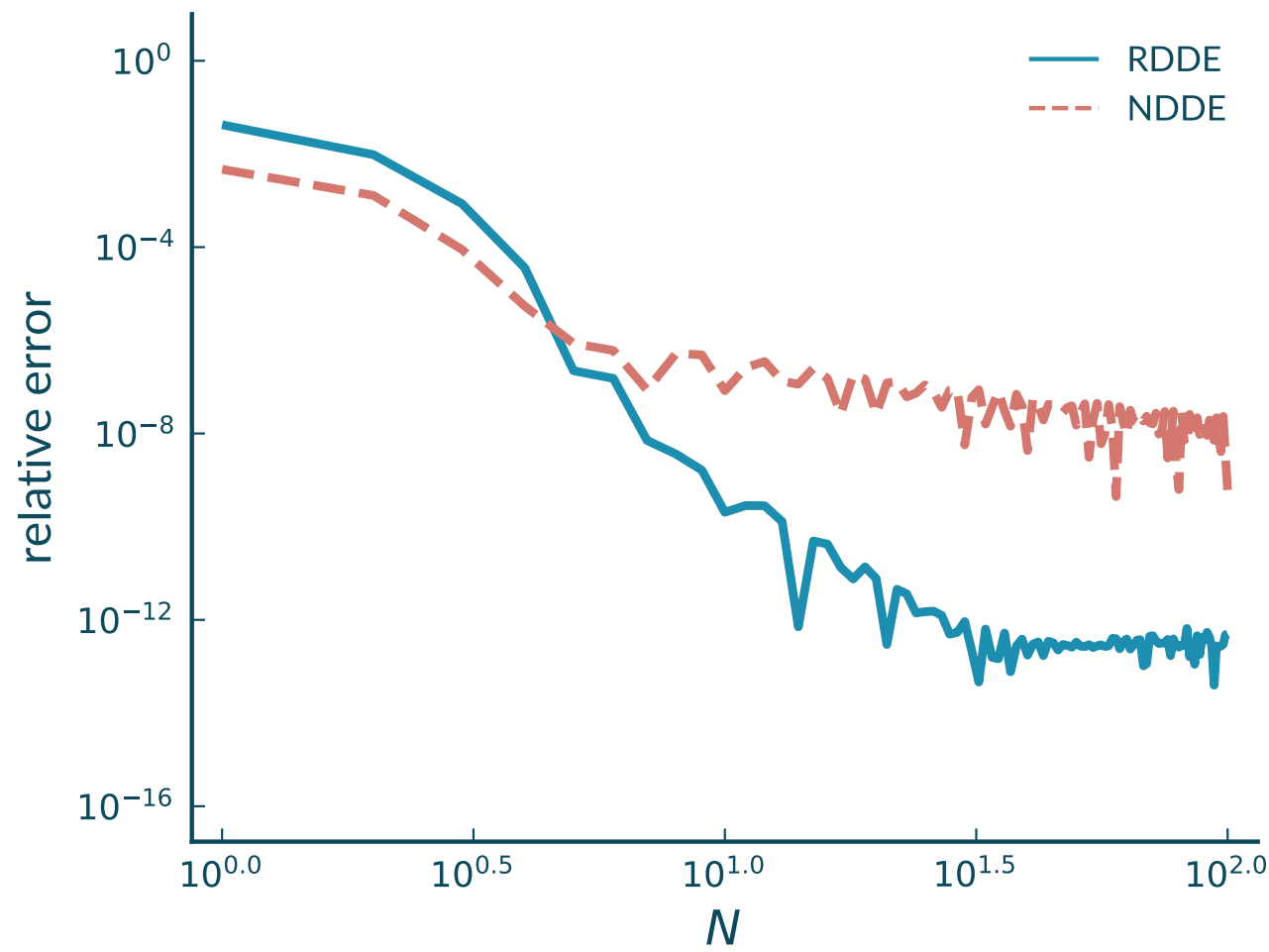
Splines



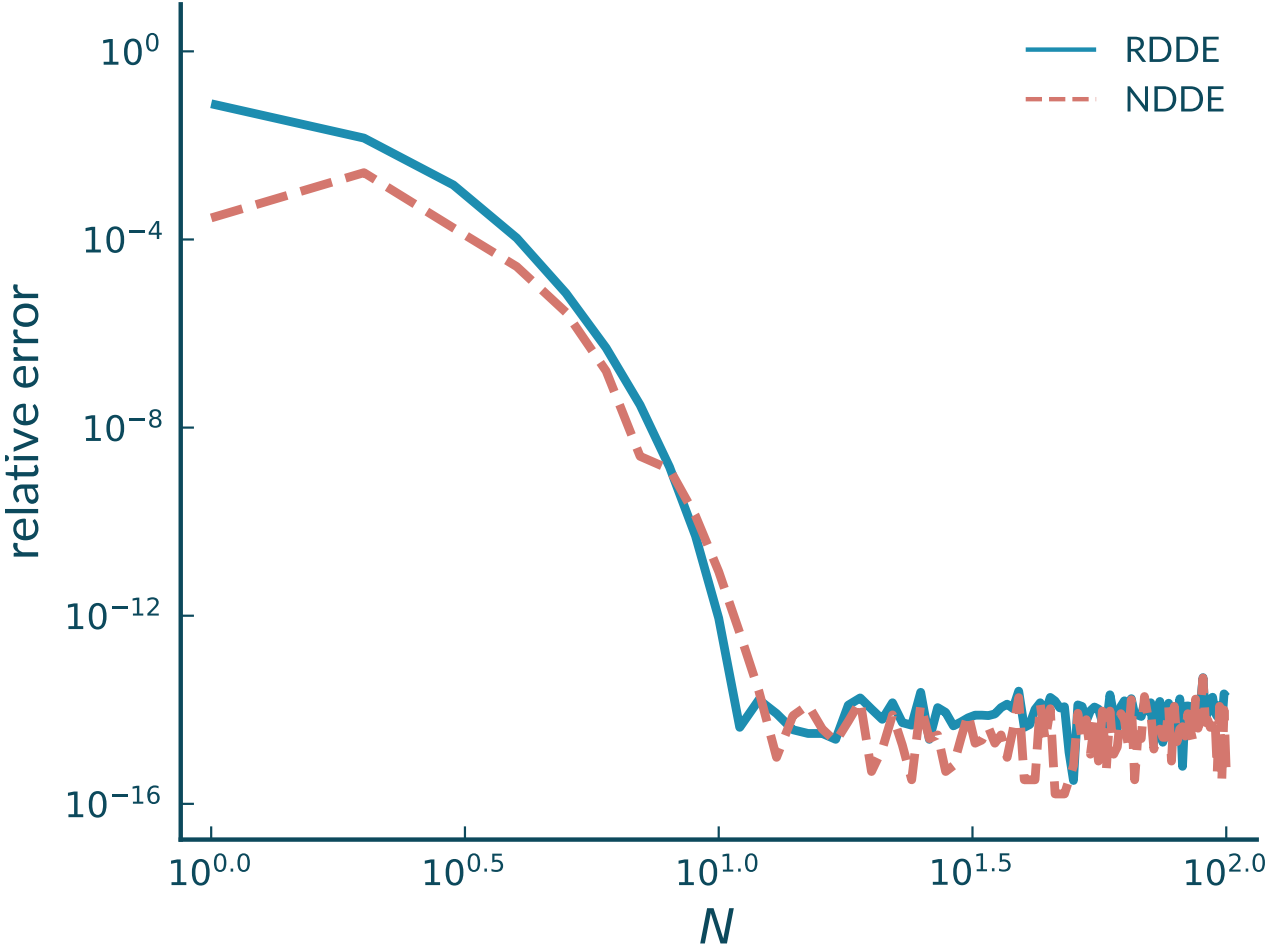
... improve convergence



... and can go further



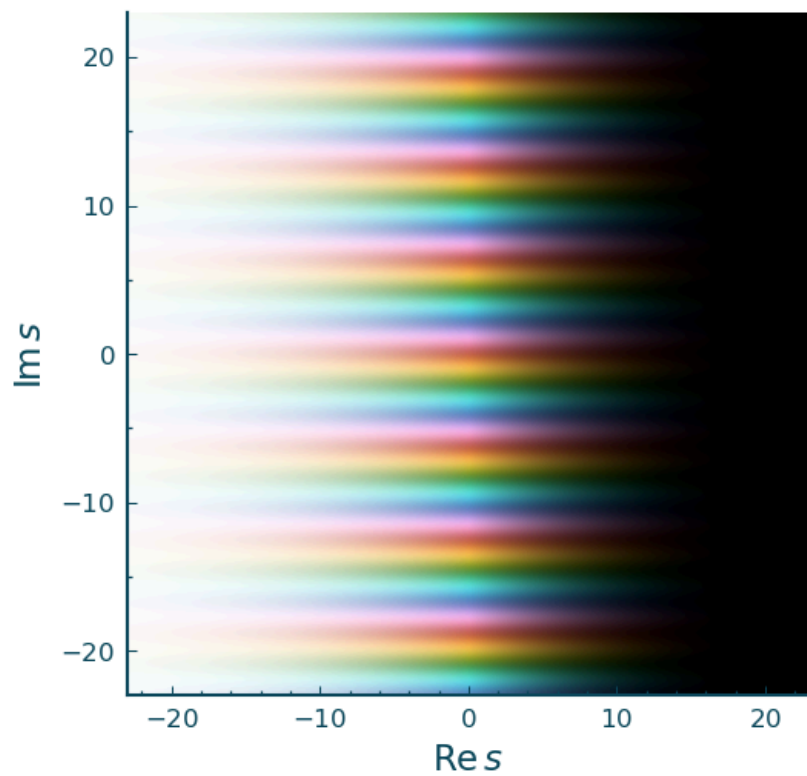
... and further still in some cases



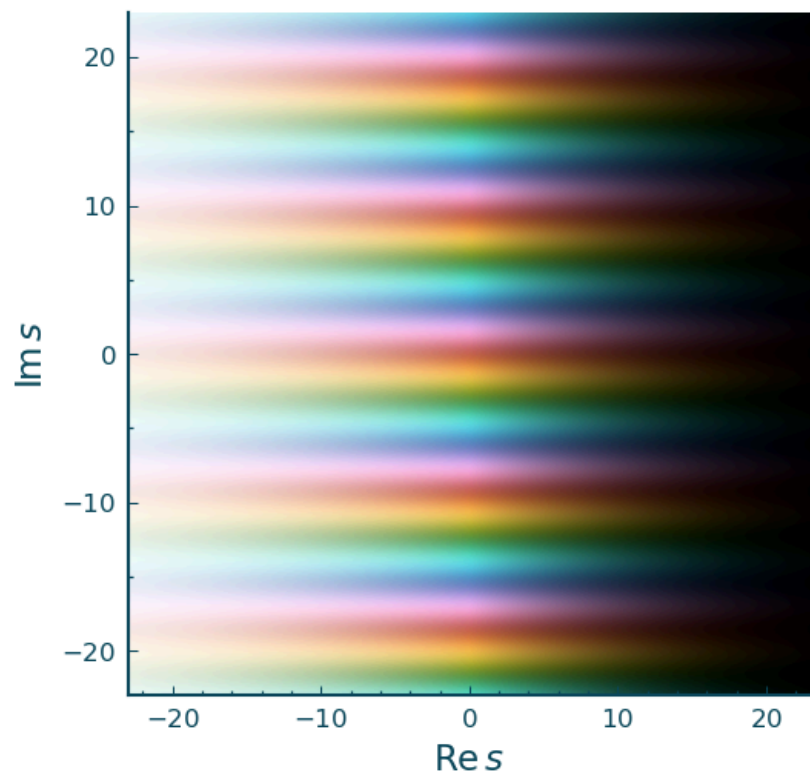
Proof of convergence

In Laplace domain

$G(s)$ contains $e^{-\theta s}$.



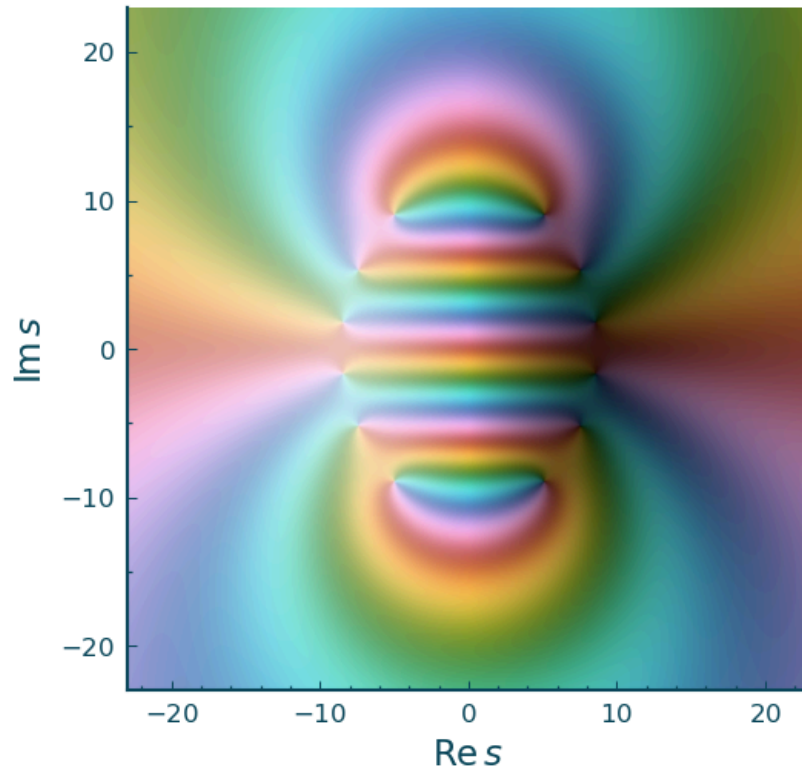
$$\theta = \tau_2$$



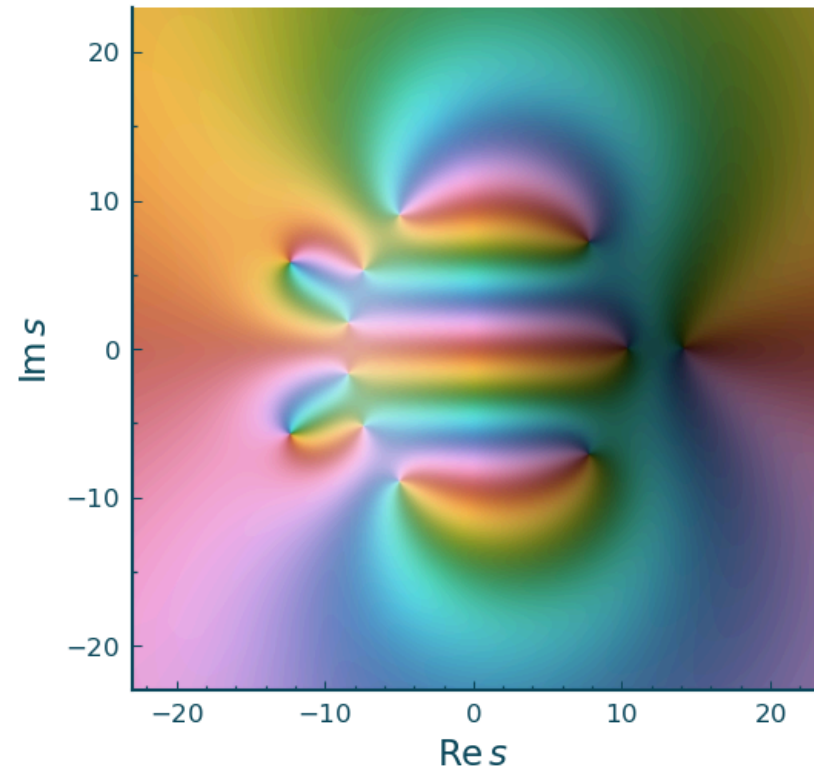
$$\theta = \tau_1 < \tau_2$$

In Laplace domain

$G(s)$ contains $e^{-\theta s} \implies G_N(s) = G(s)$ with $e^{-\theta s}$ replaced by $r_N(s, -\theta)$.



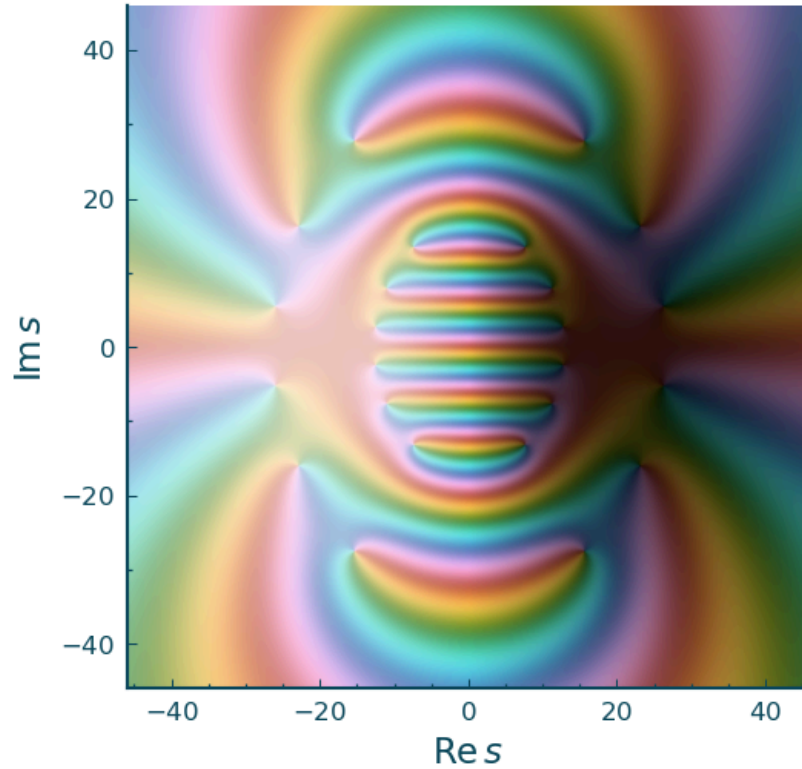
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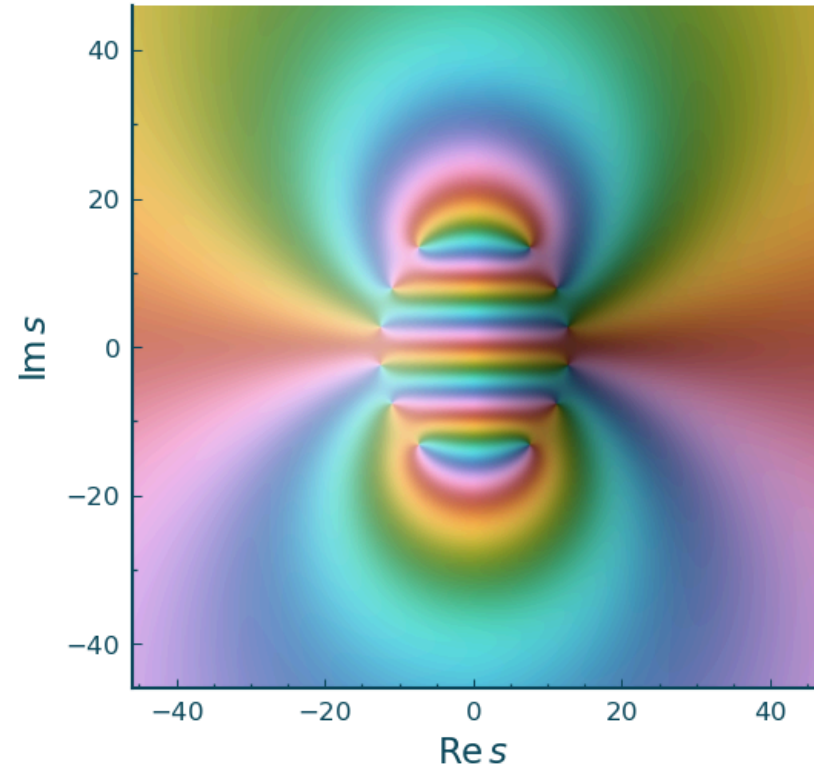
$$\theta = \tau_1 < \tau_2$$

In Laplace domain (spline)

$G(s)$ contains $e^{-\theta s} \implies G_N(s) = G(s)$ with $e^{-\theta s}$ replaced by $r_N(s, -\theta)$.



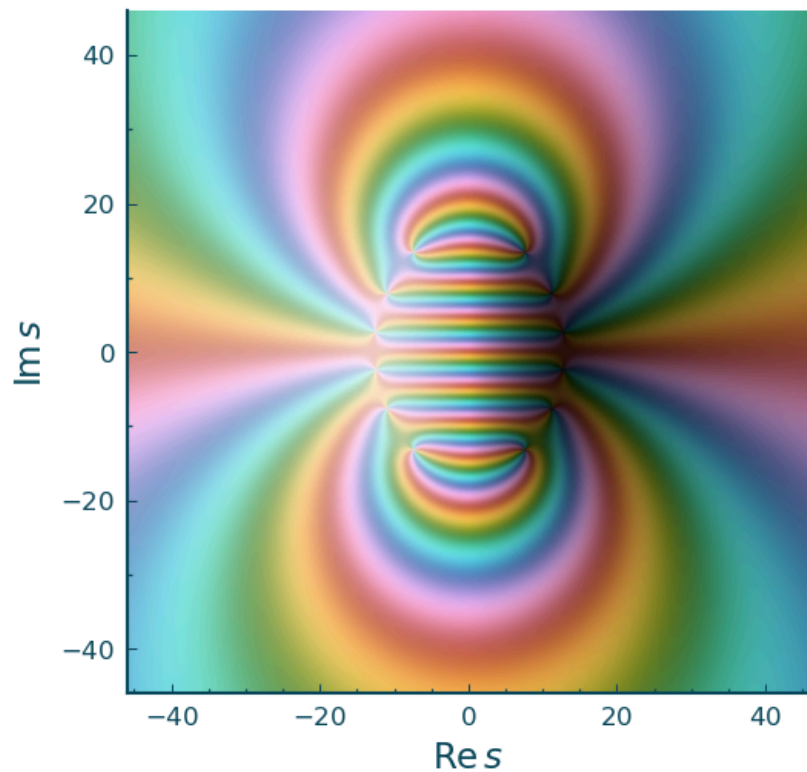
$$\theta = \tau_2$$



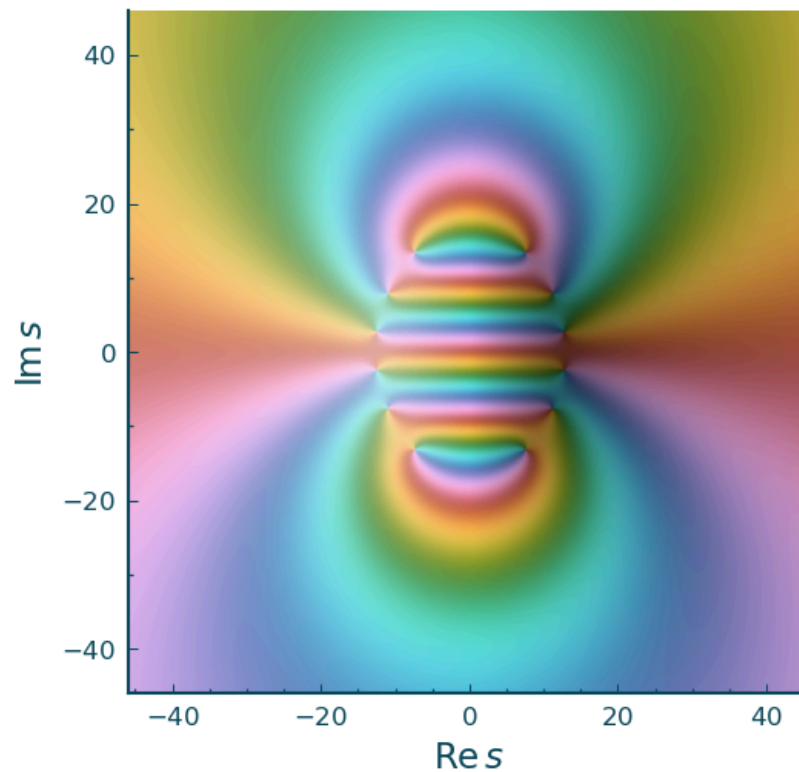
$$\theta = \tau_1 < \tau_2$$

In Laplace domain (spline, special case)

$G(s)$ contains $e^{-\theta s} \implies G_N(s) = G(s)$ with $e^{-\theta s}$ replaced by $r_N(s, -\theta)$.



$$\theta = \tau_2$$



$$\theta = \tau_1 < \tau_2$$

Idea of proof

Good approximation of $e^{-\theta s}$

$\implies$ good approximation of G

$\implies$ convergence to $\|G\|_{H^2}$.

Convergence H^2 -norm

If discretization preserves stability and satisfies some technical conditions:

$$\|G_N\|_{H^2} \rightarrow \|G\|_{H^2}.$$

Preservation of stability

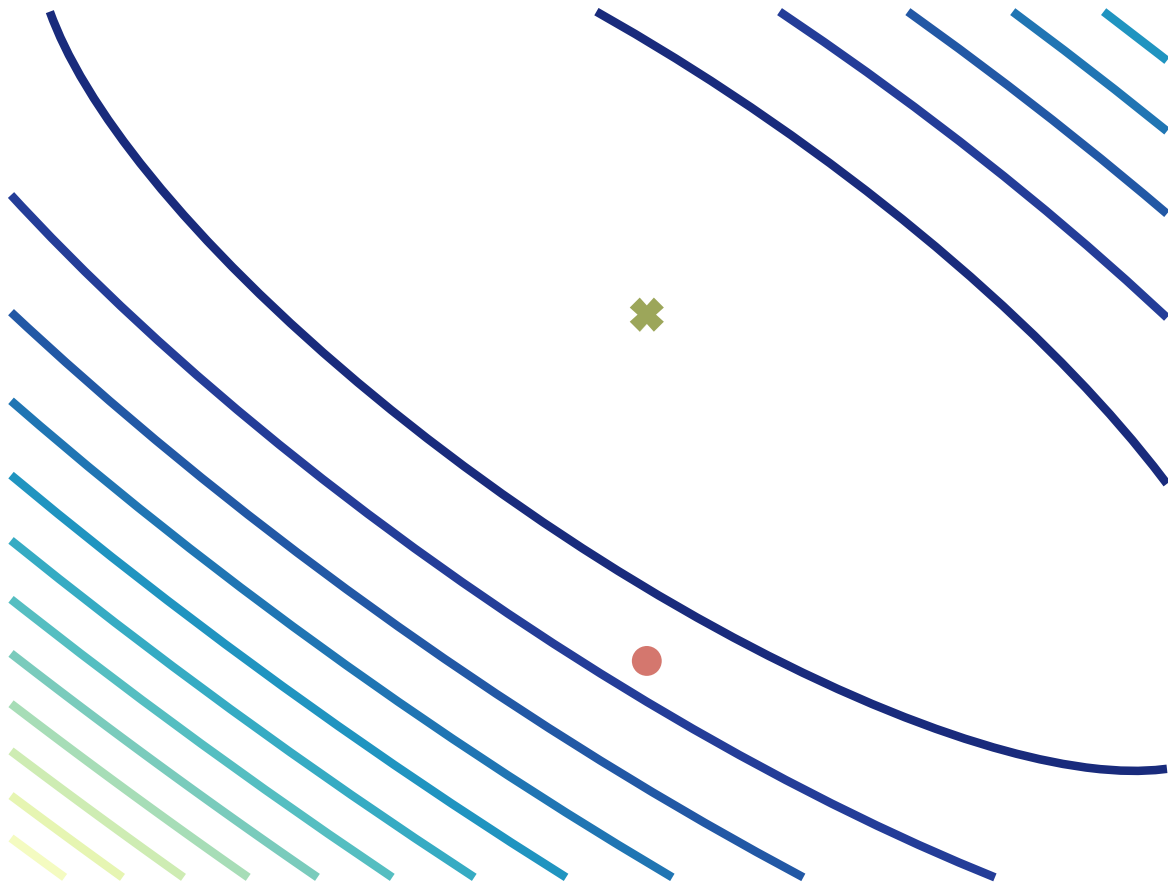
When using a Legendre spline stability is preserved.

Convergence H^2 -norm (Legendre)

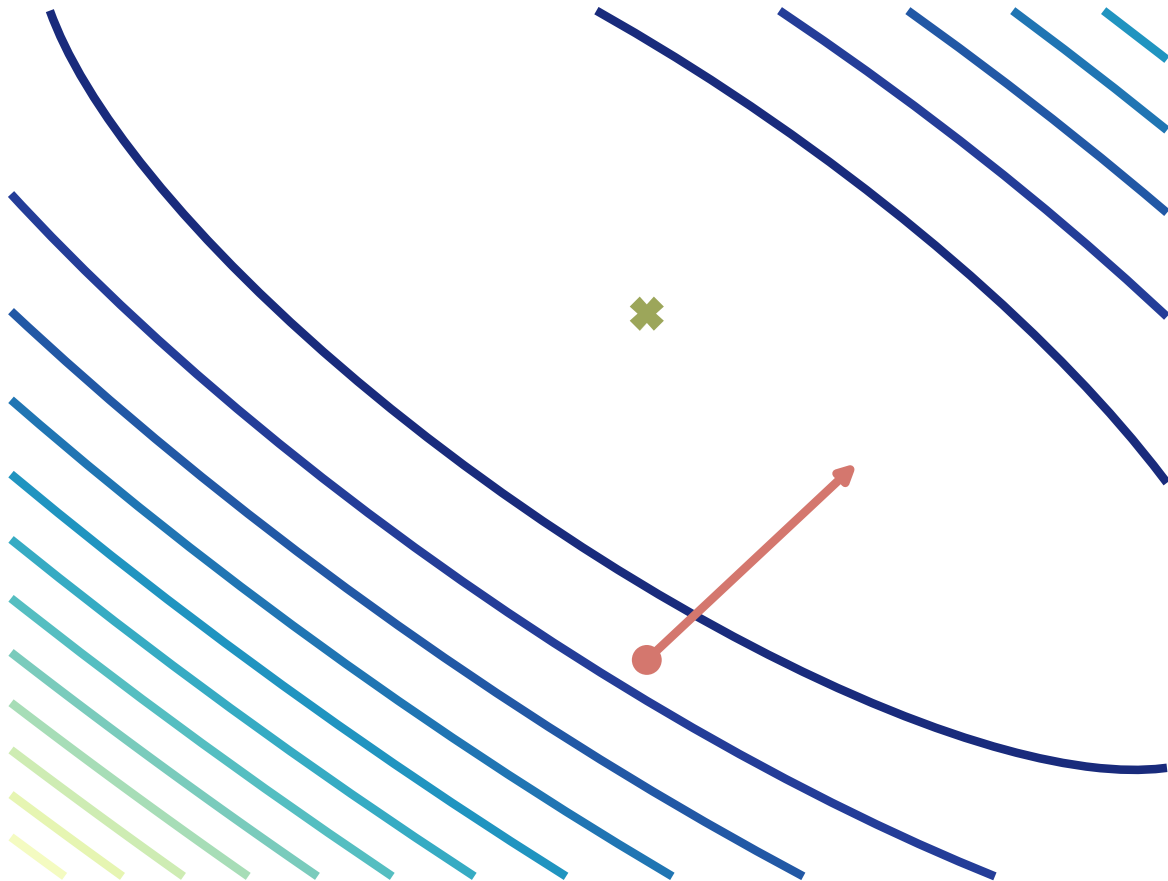
$\|G_N\|_{H^2} \rightarrow \|G\|_{H^2}$ when using Legendre splines.

H^2 -optimal synthesis

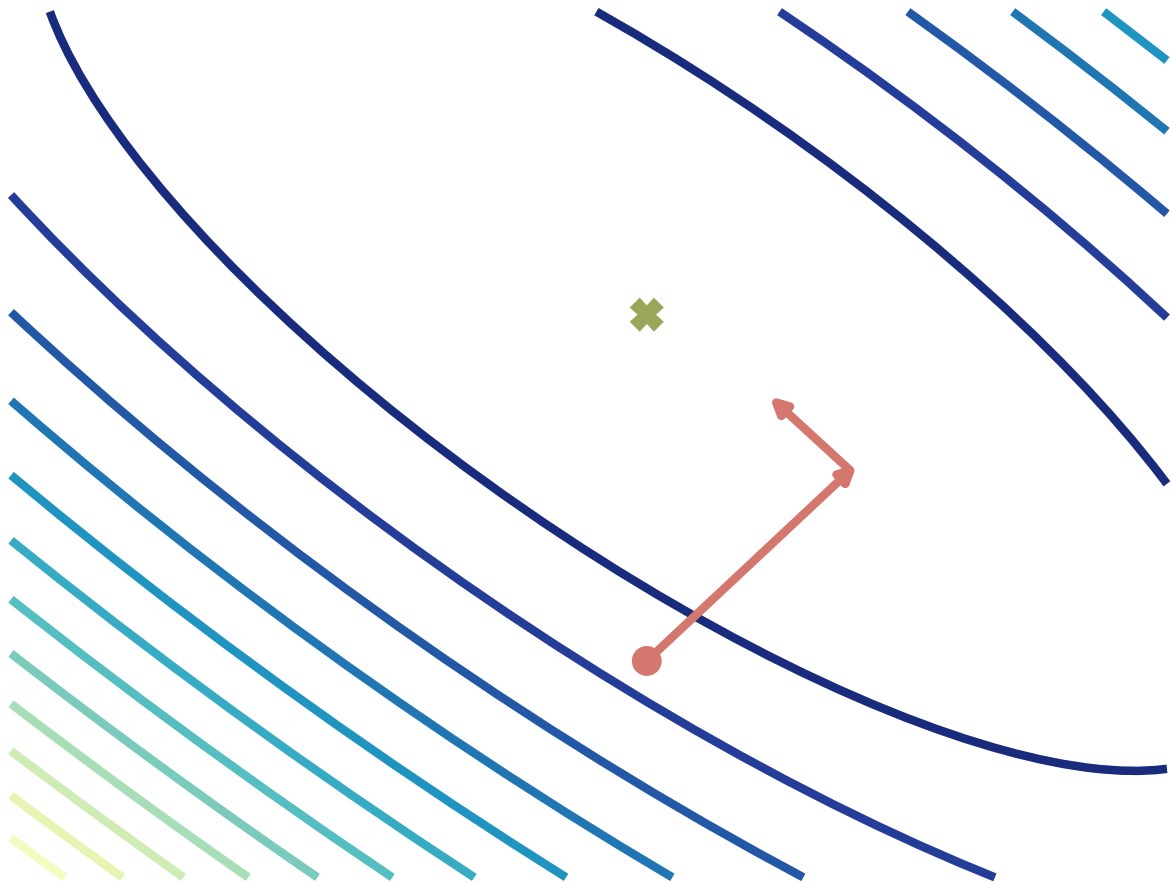
Gradient based optimization



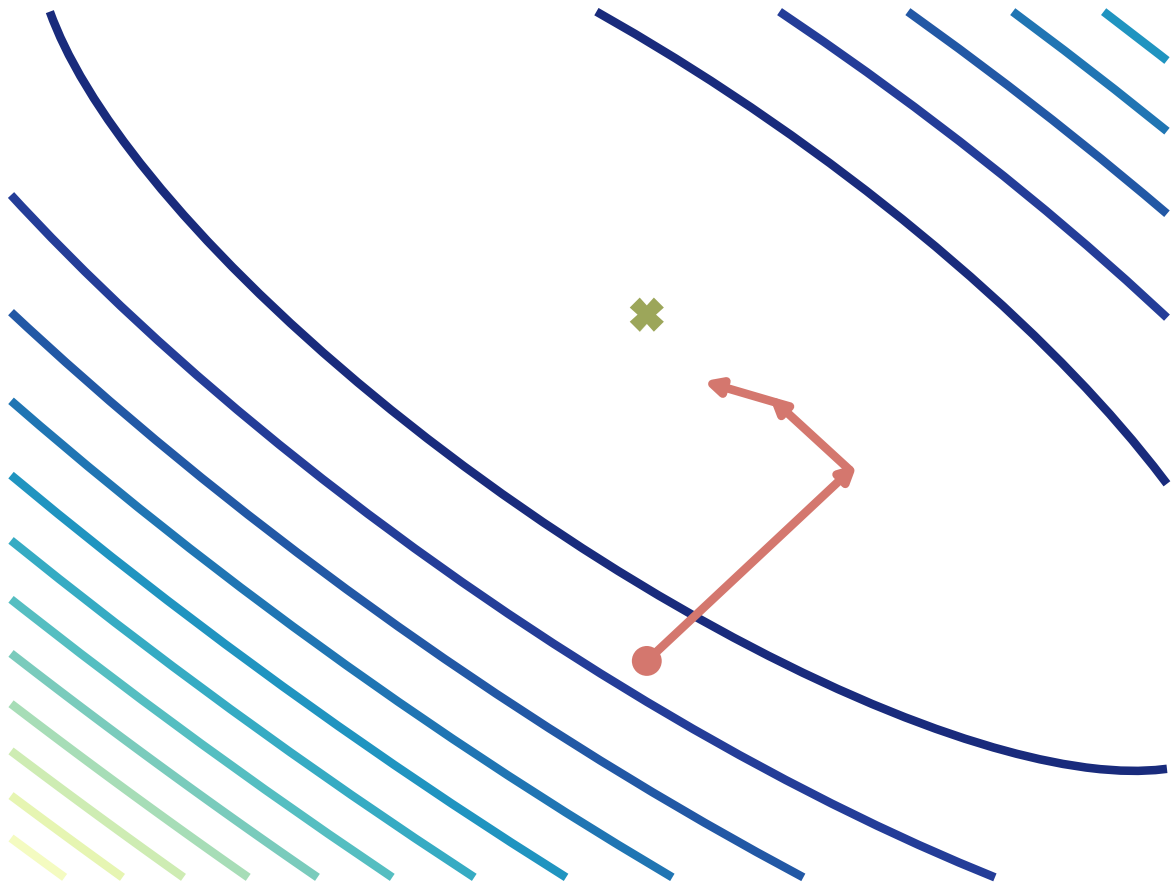
Gradient based optimization



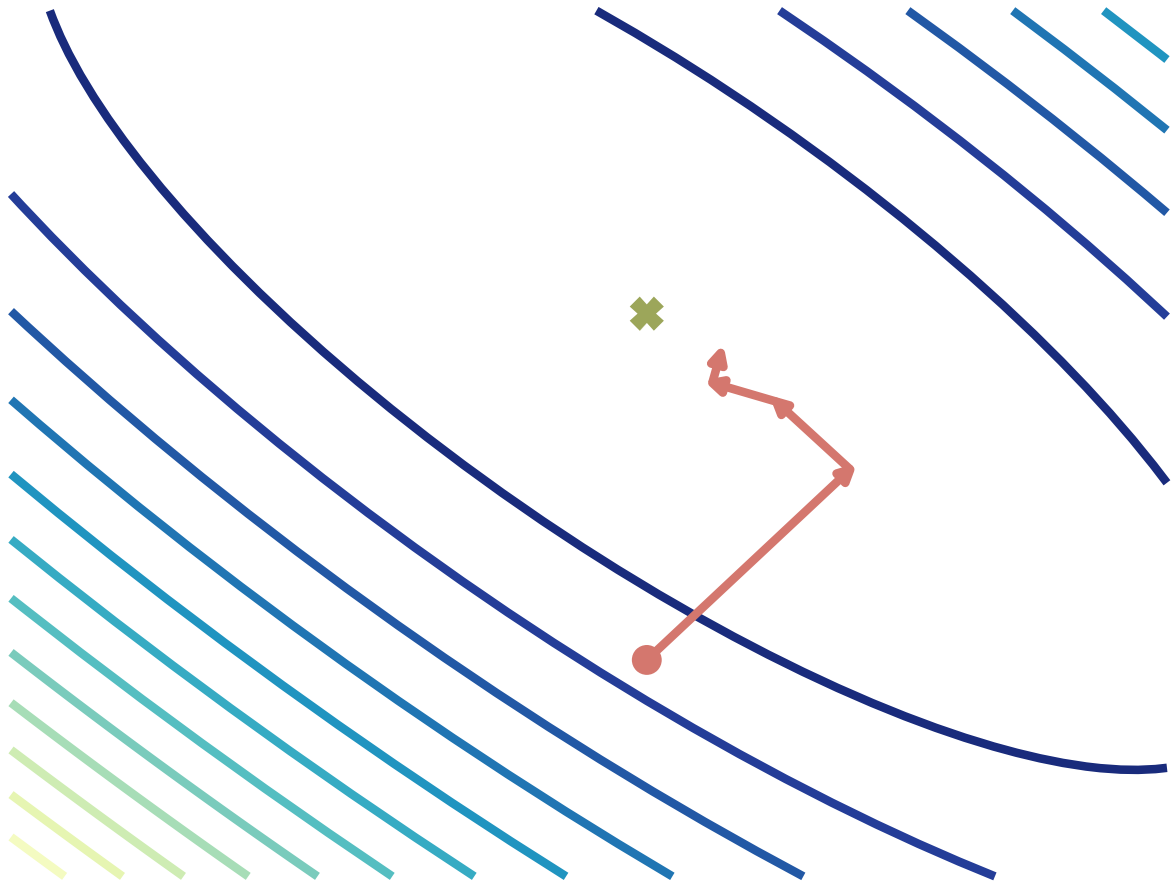
Gradient based optimization



Gradient based optimization



Gradient based optimization



Gradient based H^2 -norm optimization

$$\frac{d\|G_N\|_{H^2}^2}{dA_k} = 2 \begin{pmatrix} I_n \\ 0 \end{pmatrix}^T \mathcal{D}_S^T [\varepsilon_{-\tau_k}]^T,$$

$$\frac{d\|G_N\|_{H^2}^2}{d\tau_k} = -2 \operatorname{tr} \left(A_k [\varepsilon_{-\tau_k} \mathcal{D}] \mathcal{D}_S \begin{pmatrix} I_n \\ 0 \end{pmatrix} \right) \quad \text{for } k < m,$$

$$\frac{d\|G_N\|_{H^2}^2}{d\tau_m} = -2 \operatorname{tr} \left(\frac{1}{\tau_m} [\mathcal{D}] \mathcal{D}_S \begin{pmatrix} 0 \\ I_{nN} \end{pmatrix} \right) - \sum_{k=0}^{m-1} \frac{\tau_k}{\tau_m} \frac{d\|G_N\|_{H^2}^2}{d\tau_k},$$

$$\frac{d\|G_N\|_{H^2}^2}{dB} = 2 \begin{pmatrix} I_n \\ 0 \end{pmatrix}^T Q_U^T \tilde{B}, \quad \text{and} \quad \frac{d\|G_N\|_{H^2}^2}{dC} = 2 \tilde{C} P_V^T [\varepsilon_0]^T,$$

where $Q_U = Q(U^\perp - \mathcal{A}_{12} \mathcal{A}_{22}^{-1} U)$, $P_V = (V^\perp - V \mathcal{A}_{22}^{-1} \mathcal{A}_{21}) P$,
 $W_{\mathcal{A}} = V \mathcal{A}_{22}^{-1} U$, and $\mathcal{D}_S = P_V \mathcal{E}_{11}^T Q_U - W_{\mathcal{A}} \mathcal{B}_N \tilde{B}^T Q_U - P_V \tilde{C}^T C_N W_{\mathcal{A}}$.

Gradient based H^2 -norm optimization

Works.

What did I do?

- Extend to include algebraic constraints.
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