KU LEUVEN

Computing the $\mathscr{H}_2$-norm of Delay Differential Algebraic Systems

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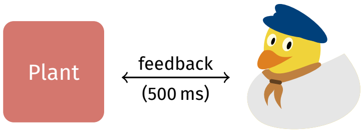
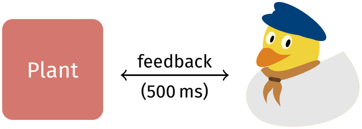
NUMA Section, Department of Computer Science

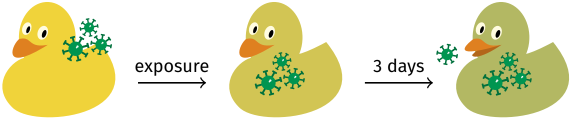
Why Delay?



(More examples: Sipahi et al. [1].)

Why Delay?





(More examples: Sipahi et al. [1].)

Delay Differential System

DDE State Space

$$\dot{\mathbf{x}}(t) = \quad A \;\mathbf{x}(t \quad) + B\mathbf{u}(t),$$

$$\mathbf{y}(t) = C\mathbf{x}(t).$$

Transfer Function

$$H(s) = C\left(sI - \quad A \quad\right)^{-1} B.$$

Delay Differential System

DDE State Space

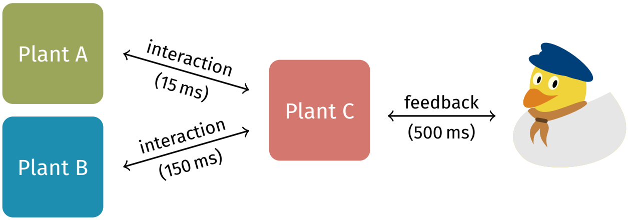
$$\dot{\mathbf{x}}(t) = \sum_{k=0}^{m} A_k \mathbf{x}(t - \tau_k) + B\mathbf{u}(t),$$
$$\mathbf{y}(t) = C\mathbf{x}(t),$$

where $0 \leq \tau_0 < \tau_1 < \cdots < \tau_m < +\infty$.

Transfer Function

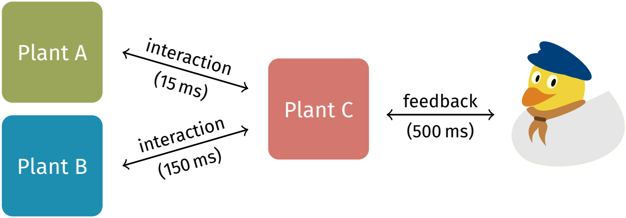
$$H(s) = C\left(sI - \sum_{k=0}^{m} A_k e^{-\tau_k s}\right)^{-1} B.$$

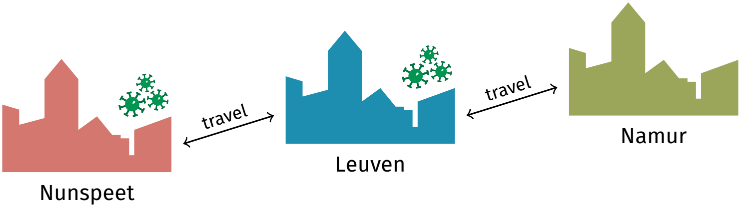
Why Algebraic?



(More motivation: Gumussoy and Michiels [2].)

Why Algebraic?





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Delay Differential Algebraic System

DDAE State Space

$$\dot{\mathbf{x}}(t) = \sum_{k=0}^{m} A_k \mathbf{x}(t - \tau_k) + B\mathbf{u}(t),$$
$$\mathbf{y}(t) = C\mathbf{x}(t),$$

where $0 \leq \tau_0 < \tau_1 < \cdots < \tau_m < +\infty$.

Transfer Function

$$H(s) = C\left(sI - \sum_{k=0}^{m} A_k e^{-\tau_k s}\right)^{-1} B.$$

Delay Differential Algebraic System

DDAE State Space

$$
\begin{aligned}
E\dot{\mathbf{x}}(t) &= \sum_{k=0}^{m} A_k \mathbf{x}(t - \tau_k) + B\mathbf{u}(t), \\
\mathbf{y}(t) &= C\mathbf{x}(t),
\end{aligned}
$$

where $0 \leq \tau_0 < \tau_1 < \cdots < \tau_m < +\infty$, and E, in general, singular.

Transfer Function

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H(s) = C\left(sE - \sum_{k=0}^{m} A_k e^{-\tau_k s}\right)^{-1} B.
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where $0 \leq \tau_0 < \tau_1 < \cdots < \tau_m < +\infty$, and E, in general, singular.

(Further assume causality and at most differentiation index 1.)

Transfer Function

$$H(s) = C\left(sE - \sum_{k=0}^{m} A_k e^{-\tau_k s}\right)^{-1} B.$$

What is the energy of
the impulse response?

Why the $\mathscr{H}_2$-norm?

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What is the steady-state power of
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the output response to unit white noise?

(Compare: $\mathscr{H}_\infty$-norm is the maximal amplification.)

The $\mathscr{H}_2$-norm

Definition

For an exponentially stable system

$$\|H\|_{\mathscr{H}_2} = \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \operatorname{Tr}(H(i\omega)^* H(i\omega)) \, \mathrm{d}\omega\right)^{\frac{1}{2}},$$

else

$$\|H\|_{\mathscr{H}_2} = \infty.$$

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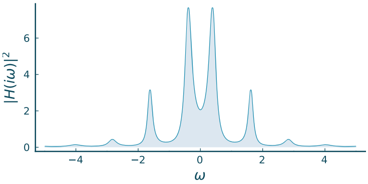

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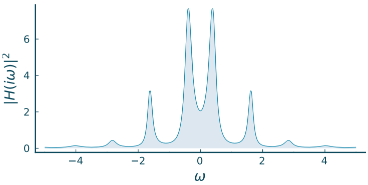

Finite when the system is stable

The $\mathscr{H}_2$-norm

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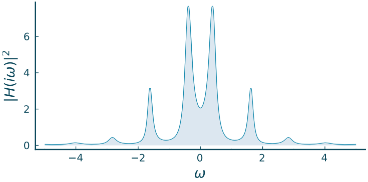

Finite when the system is stable *and has no feedthrough.*

Challenges of DDAEs

- Hidden feedthrough. *E.g.*

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \dot{\mathbf{x}}(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 1 \\ -2 \end{pmatrix} u(t),$$

$$y(t) = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x}(t).$$

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$$y(t) = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x}(t),$$

$$\implies \quad \begin{aligned} \dot{x}_1(t) &= x_1(t) + u(t), \\ y(t) &= x_1(t) + 2u(t). \end{aligned}$$

Challenges of DDAEs

- Hidden feedthrough.
- $H(s)$ usually has infinitely many poles in $\mathbb{C}^-$. *E.g.*

$$\begin{aligned} \dot{x}(t) &= -x(t-1) + u(t), \\ y(t) &= x(t). \end{aligned}$$

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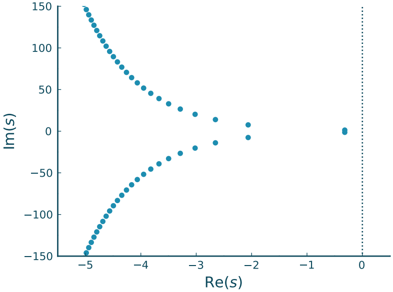
$$H(s) = (s + e^{-s})^{-1}$$
$$\implies \text{poles at } s = -\ln|s| + i(\arg s + (2k+1)\pi) \quad \forall k \in \mathbb{Z}.$$



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- Hidden feedthrough.
- $H(s)$ usually has infinitely many poles in $\mathbb{C}^-$.
- Sometimes even vertical chains. *E.g.*

$$
\begin{aligned}
\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \dot{\mathbf{x}}(t) &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 0 & -1/2 \\ 0 & 0 \end{pmatrix} \mathbf{x}(t-1) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t), \\
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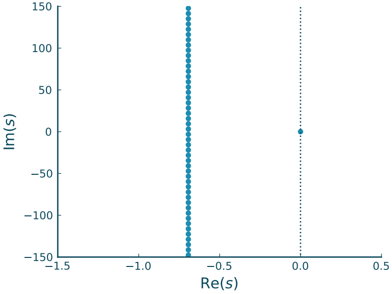
$$\implies \text{poles at } s = 0 \text{ and } s = -\ln 2 + i(2k+1)\pi \quad \forall k \in \mathbb{Z}.$$



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- Instability after infinitesimal perturbation. *E.g.*

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \dot{\mathbf{x}}(t) = \begin{pmatrix} -1/2 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 0 & 3/4 \\ 0 & 0 \end{pmatrix} \mathbf{x}(t-1) + \begin{pmatrix} 0 & -1/2 \\ 0 & 0 \end{pmatrix} \mathbf{x}(t-2).$$

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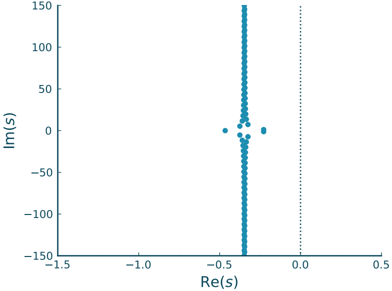

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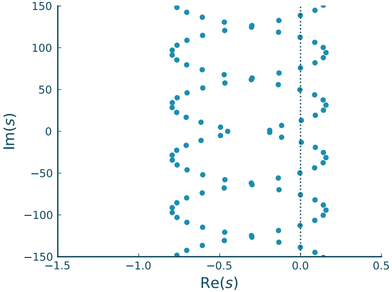

Challenges of DDAEs

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  $\implies$ *strong* stability. *E.g.*

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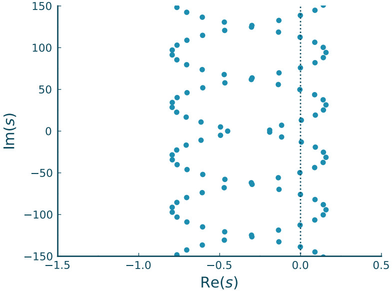

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- $H(s)$ usually has infinitely many poles in $\mathbb{C}^-$.
- Sometimes even vertical chains.
- Instability after infinitesimal perturbation
  $\Longrightarrow$ *strong* stability.
- Feedthrough after infinitesimal perturbation. *E.g.*

$$
\begin{aligned}
\mathbf{0}\,\dot{\mathbf{x}}(t) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{x}(t) - \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mathbf{x}(t-\tau_1) \\
&\quad - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mathbf{x}(t-\tau_2) - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mathbf{x}(t-\tau_3) - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u(t), \\
y(t) &= \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix} \mathbf{x}(t).
\end{aligned}
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$$y(t) = u(t - (\tau_1 + \tau_2)) - u(t - \tau_3).$$

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  $\implies$ *strong* $\mathscr{H}_2$-norm. *E.g.*

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  $\implies$ *strong* $\mathscr{H}_2$-norm.

Computing the $\mathscr{H}_2$-norm

1. Check for finiteness strong $\mathscr{H}_2$-norm.
    - Strong stability
      $\implies$ Michiels [3].
    - Feedthrough after infinitesimal perturbation
      $\implies$ Mattenet et al. [4].

Computing the $\mathscr{H}_2$-norm

1. Check for finiteness strong $\mathscr{H}_2$-norm.
2. Approximate DDAE by DAE using pseudospectral discretization.

$$\begin{pmatrix} \dot{\varphi}_1(t) \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} \\ \mathscr{A}_{21} & \mathscr{A}_{22} \end{pmatrix} \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix} + \begin{pmatrix} \mathscr{B}_1 \\ \mathscr{B}_2 \end{pmatrix} \mathbf{u}(t),$$

$$\mathbf{y}(t) \approx \begin{pmatrix} \mathscr{C}_1 & \mathscr{C}_2 \end{pmatrix} \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix}.$$

(See Breda et al. [5].)

Computing the $\mathscr{H}_2$-norm

1. Check for finiteness strong $\mathscr{H}_2$-norm.
2. Approximate DDAE by DAE using pseudospectral discretization.
3. DAE to ODE by eliminating algebraic part.

$$\begin{aligned}\dot{\varphi}_1(t) &= \tilde{A}\varphi_1(t) + \tilde{B}\mathbf{u}(t),\\ \mathbf{y}(t) &\approx \tilde{C}\varphi_1(t) + \tilde{D}\mathbf{u}(t),\end{aligned}$$

with $\tilde{A} = \mathscr{A}_{11} - \mathscr{A}_{12}\mathscr{A}_{22}^{-1}\mathscr{A}_{21}$, $\quad \tilde{B} = \mathscr{B}_1 - \mathscr{A}_{12}\mathscr{A}_{22}^{-1}\mathscr{B}_2$,

$\tilde{C} = \mathscr{C}_1 - \mathscr{C}_2\mathscr{A}_{22}^{-1}\mathscr{A}_{21}$, and $\quad \tilde{D} = -\mathscr{C}_2\mathscr{A}_{22}^{-1}\mathscr{B}_2$.

Computing the $\mathscr{H}_2$-norm

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Theorem

Under fairly mild conditions on the used basis, $\mathscr{A}_{22}$ is invertible and *no feedthrough is introduced*, if the original system satisfies step 1.

Computing the $\mathscr{H}_2$-norm

1. Check for finiteness strong $\mathscr{H}_2$-norm.
2. Approximate DDAE by DAE using pseudospectral discretization.
3. DAE to ODE by eliminating algebraic part.
4. Compute the $\mathscr{H}_2$-norm of the ODE.
   4.1 Solve $V\tilde{A}^T + \tilde{A}V = -\tilde{B}\tilde{B}^T$ for V.
   4.2 Compute

$$\|H\|_{\mathscr{H}_2}^2 \approx \mathrm{Tr}\left(\tilde{C}V\tilde{C}^T\right).$$

(See Vanbiervliet et al. [6].)

Computing the $\mathscr{H}_2$-norm

1. Check for finiteness strong $\mathscr{H}_2$-norm.
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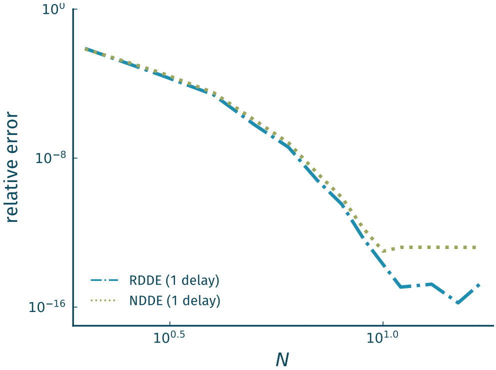
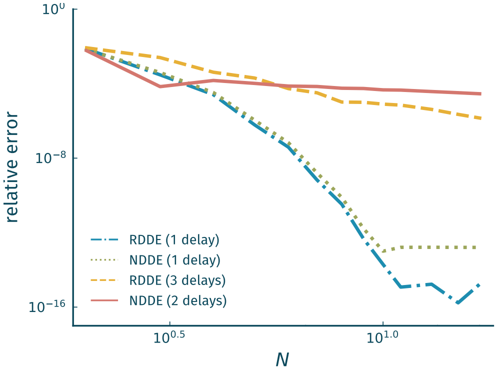
4. Compute the $\mathscr{H}_2$-norm of the ODE.

Convergence



Convergence of our method for a few easy examples.

Convergence



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Further Work

- Can we get similar convergence for multiple delays as with one?

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- How to choose N?

References

[1] Rifat Sipahi et al. "Stability and Stabilization of Systems with Time Delay". In: *IEEE Control Systems* 31.1 (2011).

[2] Suat Gumussoy and Wim Michiels. "Fixed-Order H-Infinity Control for Interconnected Systems Using Delay Differential Algebraic Equations". In: *SIAM Journal on Control and Optimization* 49.5 (2011).

[3] Wim Michiels. "Spectrum-based stability analysis and stabilisation of systems described by delay differential algebraic equations". In: *IET Control Theory & Applications* 5.16 (2011).

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[6] Joris Vanbiervliet et al. "Using spectral discretisation for the optimal $\mathscr{H}_2$ design of time-delay systems". In: *International Journal of Control* 84.2 (2011).

Contributions

- A straightforward algorithm for the $\mathscr{H}_2$-norm of DDAEs.

*Not discussed in this presentation.

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- A technique to shift poles of an unstable discretization.*

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Contributions & Further Work

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