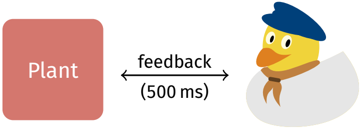
KU LEUVEN

Efficient H^2-norm Optimization of Time-Delay Systems with Algebraic Constraints

Evert Provoost and Wim Michiels

Why delay?



(More examples: Sipahi et al. (2011).)

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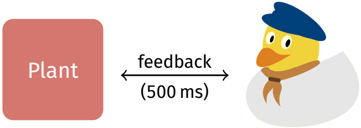

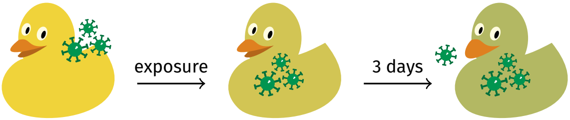

(More examples: Sipahi et al. (2011).)

Delay differential system

DDE state space

$$\dot{\mathbf{x}}(t) = \quad A \; \mathbf{x}(t \quad\quad) + B\mathbf{u}(t),$$

$$\mathbf{y}(t) = C\mathbf{x}(t).$$

Transfer function

$$G(s) = C\left(sI - \quad A \quad\quad\right)^{-1} B.$$

Delay differential system

DDE state space

$$\dot{\mathbf{x}}(t) = \sum_{k=0}^{m} A_k \mathbf{x}(t - \tau_k) + B\mathbf{u}(t),$$
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where $0 \leq \tau_0 < \tau_1 < \cdots < \tau_m < +\infty$.

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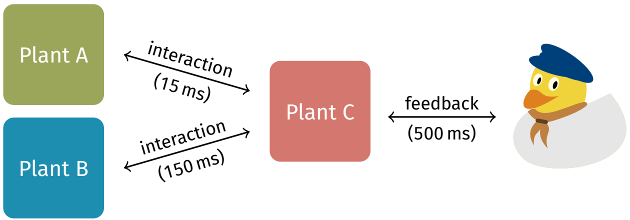
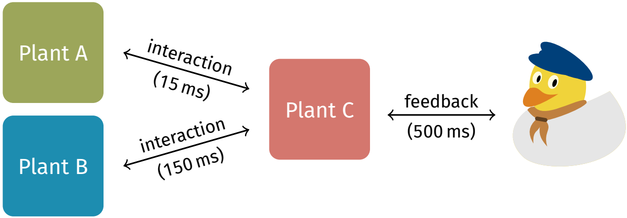
$$G(s) = C\left(sI - \sum_{k=0}^{m} A_k e^{-\tau_k s}\right)^{-1} B.$$

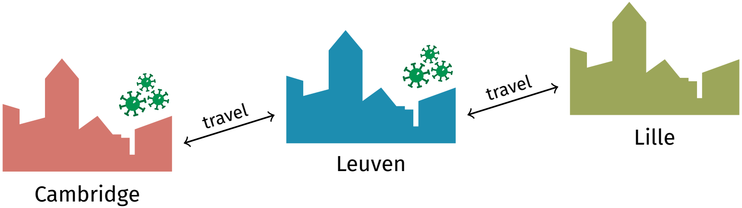
Why algebraic constraints?



(More motivation: Gumussoy and Michiels (2011).)

Why algebraic constraints?





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Delay differential algebraic system

DDAE state space

$$\dot{\mathbf{x}}(t) = \sum_{k=0}^{m} A_k \mathbf{x}(t - \tau_k) + B\mathbf{u}(t),$$
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$$G(s) = C\left(sI - \sum_{k=0}^{m} A_k e^{-\tau_k s}\right)^{-1} B.$$

Delay differential algebraic system

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$$
\begin{aligned}
E\dot{\mathbf{x}}(t) &= \sum_{k=0}^{m} A_k \mathbf{x}(t - \tau_k) + B\mathbf{u}(t), \\
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\end{aligned}
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where $0 \leq \tau_0 < \tau_1 < \cdots < \tau_m < +\infty$, and E, in general, singular.

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(Further assume causality and at most differentiation index 1.)

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$$

What is the energy of
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What is the steady-state power of
the output response to unit white noise?

Why the H^2-norm?

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What is the steady-state power of
the output response to unit white noise?

(Compare: H^∞-norm is the maximal amplification.)

The H^2-norm

Definition

For an exponentially stable system

$$\|G\|_{H^2} = \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} \|G(i\omega)\|_F^2 \,\mathrm{d}\omega\right)^{\frac{1}{2}} \quad \text{else} \quad \|G\|_{H^2} = \infty.$$

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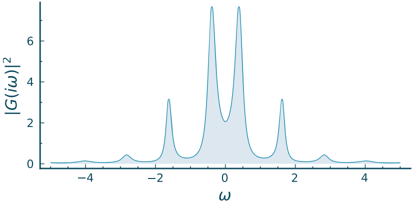

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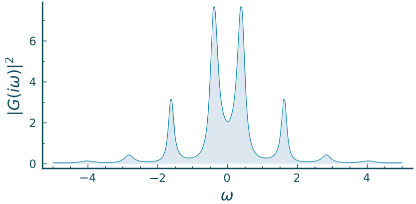

Finite when the system is stable

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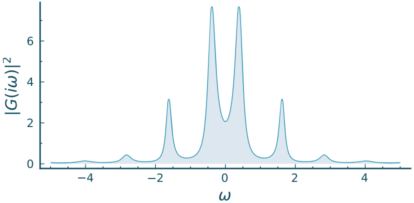

Finite when the system is stable *and has no feedthrough.*

Challenges of DDAEs

- Hidden feedthrough. E.g.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \dot{\mathbf{x}}(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 1 \\ -2 \end{pmatrix} u(t),$$
$$y(t) = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x}(t).$$

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$$y(t) = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x}(t),$$

$$\Longrightarrow \qquad \begin{aligned} \dot{x}_1(t) &= x_1(t) + u(t), \\ y(t) &= x_1(t) + 2u(t). \end{aligned}$$

Challenges of DDAEs

- Hidden feedthrough.
- $G(s)$ usually has infinitely many poles in $\mathbb{C}^-$. E.g.

$$\dot{x}(t) = -x(t-1) + u(t),$$
$$y(t) = x(t).$$

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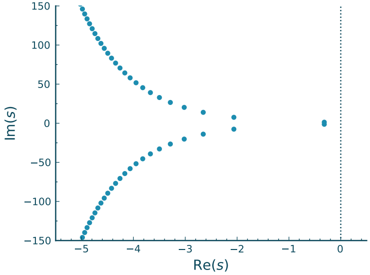
$$\implies \text{poles at } s = -\ln|s| + i(\arg s + (2k+1)\pi) \quad \forall k \in \mathbb{Z}.$$



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- Sometimes even vertical chains. E.g.

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \dot{\mathbf{x}}(t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 0 & -1/2 \\ 0 & 0 \end{pmatrix} \mathbf{x}(t-1) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t),$$
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$$\implies \quad \begin{aligned} \dot{x}_1(t) &= -\frac{1}{2}\dot{x}_1(t-1) + u(t), \\ y(t) &= x_1(t). \end{aligned}$$

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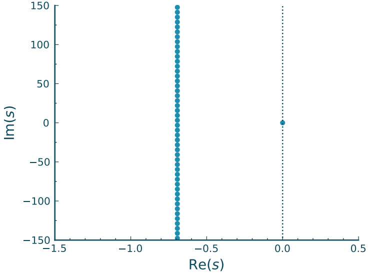

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- $G(s)$ usually has infinitely many poles in $\mathbb{C}^-$.
- Sometimes even vertical chains.
- Instability after infinitesimal perturbation. E.g.

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \dot{\mathbf{x}}(t) = \begin{pmatrix} -1/2 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 0 & 3/4 \\ 0 & 0 \end{pmatrix} \mathbf{x}(t-1) + \begin{pmatrix} 0 & -1/2 \\ 0 & 0 \end{pmatrix} \mathbf{x}(t-2).$$

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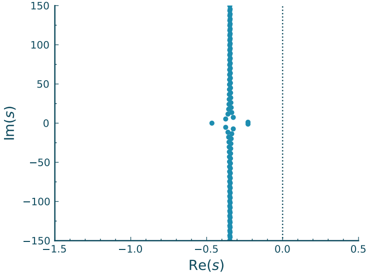

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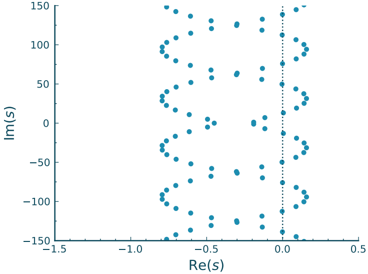

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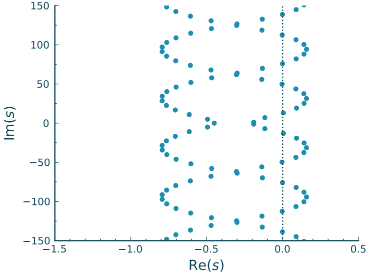

Challenges of DDAEs

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- $G(s)$ usually has infinitely many poles in $\mathbb{C}^-$.
- Sometimes even vertical chains.
- Instability after infinitesimal perturbation
  $\Longrightarrow$ *strong* stability.
- Feedthrough after infinitesimal perturbation. E.g.

$$
\begin{aligned}
\mathbf{0}\,\dot{\mathbf{x}}(t) &= \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}\mathbf{x}(t) - \begin{pmatrix} 0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}\mathbf{x}(t-\tau_1)\\
&\quad - \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix}\mathbf{x}(t-\tau_2) - \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{pmatrix}\mathbf{x}(t-\tau_3) - \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}u(t),\\
y(t) &= \begin{pmatrix} 1&1&0&0 \end{pmatrix}\mathbf{x}(t).
\end{aligned}
$$

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$$y(t) = u(t - (\tau_1 + \tau_2)) - u(t - \tau_3).$$

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  $\implies$ *strong* H^2-norm. E.g.

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Computing the H^2-norm of an ODE

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t),$$
$$\mathbf{y}(t) = C\mathbf{x}(t).$$

$$\|G\|_{H^2} = \sqrt{\operatorname{tr}(CPC^T)}, \text{ where}$$

$$AP \quad + \quad PA^T = -BB^T.$$

(See Zhou et al. (1995, Lemma 4.6).)

Computing the H^2-norm of an ODE

$$
\begin{aligned}
E\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t),\\
\mathbf{y}(t) &= C\mathbf{x}(t),
\end{aligned}
$$

with E invertible.

$$\|G\|_{H^2} = \sqrt{\operatorname{tr}(CPC^T)}, \text{ where}$$

$$APE^T + EPA^T = -BB^T.$$

(See Zhou et al. (1995, Lemma 4.6).)

Approximating the H^2-norm of a DDAE

1. Check for finiteness strong H^2-norm.
    - No instability after infinitesimal delay perturbation?
      $\implies$ Michiels (2011).
    - No feedthrough after infinitesimal delay perturbation?
      $\implies$ Mattenet et al. (2022).

Approximating the H^2-norm of a DDAE

1. Check for finiteness strong H^2-norm.
2. Approximate DDAE by DAE using spectral method.*

$$\begin{pmatrix} \mathcal{E}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \dot{\varphi}_1(t) \\ \dot{\varphi}_2(t) \end{pmatrix} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{pmatrix} \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix} + \begin{pmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{pmatrix} \mathbf{u}(t),$$

$$\mathbf{y}_N(t) = \begin{pmatrix} \mathcal{C}_1 & \mathcal{C}_2 \end{pmatrix} \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix}.$$

*See e.g. Provoost and Michiels (2024).
Here already after projection on subspaces of $\mathcal{E}_N$, such that $\mathcal{E}_{11}$ is invertible.

Approximating the H^2-norm of a DDAE

1. Check for finiteness strong H^2-norm.
2. Approximate DDAE by DAE using spectral method.
3. DAE to ODE by eliminating algebraic part.

$$\begin{aligned} \mathcal{E}_{11}\dot{\varphi}_1(t) &= \tilde{A}\varphi_1(t) + \tilde{B}\mathbf{u}(t), \\ \mathbf{y}_N(t) &= \tilde{C}\varphi_1(t) + \tilde{D}\mathbf{u}(t), \end{aligned}$$

with $\tilde{A} = \mathcal{A}_{11} - \mathcal{A}_{12}\mathcal{A}_{22}^{-1}\mathcal{A}_{21}$, $\tilde{B} = \mathcal{B}_1 - \mathcal{A}_{12}\mathcal{A}_{22}^{-1}\mathcal{B}_2$, $\tilde{C} = \mathcal{C}_1 - \mathcal{C}_2\mathcal{A}_{22}^{-1}\mathcal{A}_{21}$, and $\tilde{D} = -\mathcal{C}_2\mathcal{A}_{22}^{-1}\mathcal{B}_2$.

Approximating the H^2-norm of a DDAE

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$$\mathbf{y}_N(t) = \tilde{C}\varphi_1(t),$$

with $\tilde{A} = \mathcal{A}_{11} - \mathcal{A}_{12}\mathcal{A}_{22}^{-1}\mathcal{A}_{21}$, $\tilde{B} = \mathcal{B}_1 - \mathcal{A}_{12}\mathcal{A}_{22}^{-1}\mathcal{B}_2$,
$\tilde{C} = \mathcal{C}_1 - \mathcal{C}_2\mathcal{A}_{22}^{-1}\mathcal{A}_{21}$, and $\tilde{D} = -\mathcal{C}_2\mathcal{A}_{22}^{-1}\mathcal{B}_2 = \mathbf{0}$.

Theorem

$\mathcal{A}_{22}$ is almost always invertible and *no feedthrough is introduced*, if the original system is strongly stable and has no hidden feedthrough.

Approximating the H^2-norm of a DDAE

1. Check for finiteness strong H^2-norm.
2. Approximate DDAE by DAE using spectral method.
3. DAE to ODE by eliminating algebraic part.
4. Compute the H^2-norm of the ODE.
   4.1 Solve $\tilde{A}P\mathcal{E}_{11}^T + \mathcal{E}_{11}P\tilde{A}^T = -\tilde{B}\tilde{B}^T$ for P.
   4.2 Compute

$$\|G\|_{H^2} \approx \sqrt{\operatorname{tr}\left(\tilde{C}P\tilde{C}^T\right)}.$$

Approximating the H^2-norm of a DDAE

1. Check for finiteness strong H^2-norm.
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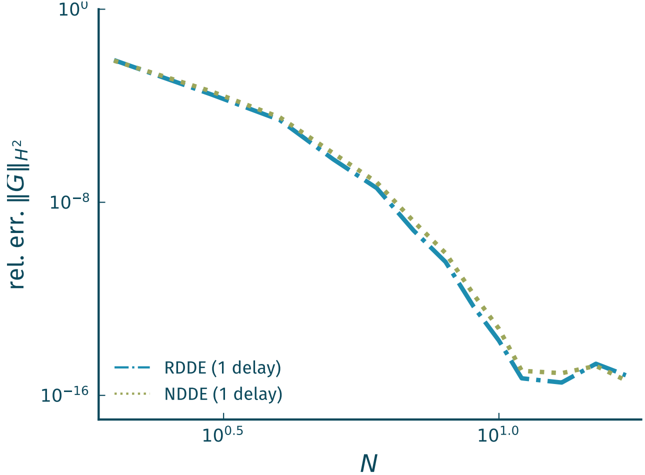
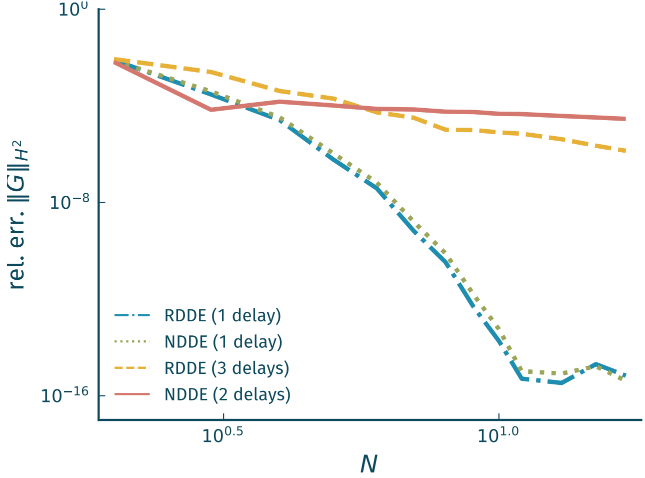
Convergence



Convergence for some simple examples using a Lanczos tau method in U_N.

Convergence



Convergence for some simple examples using a Lanczos tau method in U_N.

Computing the derivative

We can find explicit expressions using the dual Lyapunov equation

$$\tilde{A}^T Q \mathcal{E}_{11} + \mathcal{E}_{11}^T Q \tilde{A} = -\tilde{C}^T \tilde{C},$$

from a typical adjoint style method.

(Analogous to Vanbiervliet et al. (2009).)

Computing the derivative

Let $\mathcal{L}P = \tilde{A}P\mathcal{E}_{11}^T + \mathcal{E}_{11}P\tilde{A}^T$, then

$$\ell_P = \mathcal{L}P + \tilde{B}\tilde{B}^T = \mathbf{0}, \quad \text{and}$$

Computing the derivative

Let $\mathcal{L}P = \tilde{A}P\mathcal{E}_{11}^T + \mathcal{E}_{11}P\tilde{A}^T$, then

$$\ell_P = \mathcal{L}P + \tilde{B}\tilde{B}^T = \mathbf{0}, \quad \text{and}$$
$$\ell_Q = \mathcal{L}^*Q + \tilde{C}^T\tilde{C} = \mathbf{0}.$$

Computing the derivative

Let X be some variable different from P or Q, then the total derivative is

$$\begin{aligned} \mathrm{d}\ell_P &= \frac{\partial \ell_P}{\partial P}\,\mathrm{d}P + \frac{\partial \ell_P}{\partial X}\,\mathrm{d}X = \mathbf{0}, \quad \text{and} \\ \ell_Q &= \mathcal{L}^* Q + \tilde{C}^T \tilde{C} = \mathbf{0}. \end{aligned}$$

Computing the derivative

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$$\begin{aligned} &\mathcal{L}\,\mathrm{d}P + \frac{\partial \ell_P}{\partial X}\,\mathrm{d}X = \mathbf{0}, \quad \text{and} \\ &\mathcal{L}^* Q + \tilde{C}^T \tilde{C} = \mathbf{0}. \end{aligned}$$

Computing the derivative

Then,

$$
\begin{aligned}
\mathrm{d}P &= -\mathcal{L}^{-1} \frac{\partial \ell_P}{\partial X}\, \mathrm{d}X, \quad \text{and} \\
Q &= -\mathcal{L}^{-*} \tilde{C}^T \tilde{C}.
\end{aligned}
$$

Computing the derivative

From $Q = Q^T$ we have,

$$\tilde{C}^T \tilde{C} \, \mathrm{d}P = Q \frac{\partial \ell_P}{\partial X} \, \mathrm{d}X \,, \quad \text{and}$$

Computing the derivative

From $Q = Q^T$ and $P = P^T$ we have,

$$\tilde{C}^T\tilde{C}\,\mathrm{d}P = Q\frac{\partial \ell_P}{\partial X}\,\mathrm{d}X\,, \quad \text{and}$$
$$\tilde{B}\tilde{B}^T\,\mathrm{d}Q = P\frac{\partial \ell_Q}{\partial X}\,\mathrm{d}X\,.$$

Computing the derivative

From $\|G_N\|_{H^2}^2 = \mathrm{tr}\left(\tilde{C}P\tilde{C}^T\right) = \mathrm{tr}\left(\tilde{B}^TQ\tilde{B}\right)$,

Computing the derivative

From $\|G_N\|_{H^2}^2 = \operatorname{tr}\left(\tilde{C}P\tilde{C}^T\right) = \operatorname{tr}\left(\tilde{B}^TQ\tilde{B}\right)$,

$$\begin{aligned}
\mathrm{d}\|G_N\|_{H^2}^2 &= \operatorname{tr}\left(Q\frac{\partial \ell_P}{\partial X}\,\mathrm{d}X\right) \\
&= \operatorname{tr}\left(P\frac{\partial \ell_Q}{\partial X}\,\mathrm{d}X\right).
\end{aligned}$$

Computing the derivative

After many tedious but simple steps and from $\mathrm{d}f = \mathrm{tr}\left(Y^T \mathrm{d}X\right) \implies \frac{\mathrm{d}f}{\mathrm{d}X} = Y$,

(See Magnus and Neudecker (1985).)

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$$\frac{\mathrm{d}\|G_N\|_{H^2}^2}{\mathrm{d}A_k} = 2\begin{pmatrix} I_n \\ \mathbf{0} \end{pmatrix}^T \left(Q_U \mathcal{E}_{11} P_V - Q_U \tilde{B} \mathcal{B}_N^T W_{\mathcal{A}}^T - W_{\mathcal{A}}^T \mathcal{C}_N^T \tilde{C} P_V\right)[\varepsilon_{-\tau_k}]^T,$$

$$\frac{\mathrm{d}\|G_N\|_{H^2}^2}{\mathrm{d}\tau_k} = -2\operatorname{tr}\left(\begin{pmatrix} I_n \\ \mathbf{0} \end{pmatrix}^T \left(Q_U \mathcal{E}_{11} P_V - Q_U \tilde{B} \mathcal{B}_N^T W_{\mathcal{A}}^T - W_{\mathcal{A}}^T \mathcal{C}_N^T \tilde{C} P_V\right)[\varepsilon_{-\tau_k} \mathcal{D}]^T A_k^T\right),$$

$$\frac{\mathrm{d}\|G_N\|_{H^2}^2}{\mathrm{d}B} = 2\begin{pmatrix} I_n \\ \mathbf{0} \end{pmatrix}^T Q_U \tilde{B}, \quad \text{and} \quad \frac{\mathrm{d}\|G_N\|_{H^2}^2}{\mathrm{d}C} = 2\tilde{C} P_V [\varepsilon_0]^T,$$

where $Q_U = (U^\perp - \mathcal{A}_{12}\mathcal{A}_{22}^{-1}U)^T Q$, $P_V = P(V^\perp - V\mathcal{A}_{22}^{-1}\mathcal{A}_{21})^T$, and $W_{\mathcal{A}} = V\mathcal{A}_{22}^{-1}U$.

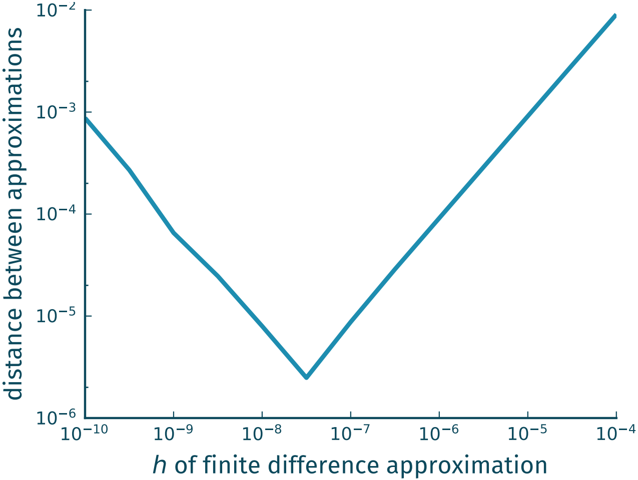
(See Magnus and Neudecker (1985).)

Computing the derivative

By only solving one additional Lyapunov equation

$\implies$ the derivative with respect to every parameter.

Validation of derivatives



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References

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