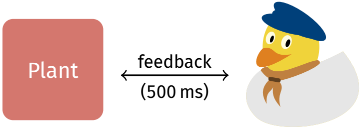
KU LEUVEN

The Lanczos Tau Framework for Time-Delay Systems

Evert Provoost and Wim Michiels

Preprint of same title available at https://arxiv.org/abs/2403.03895.

Why delay?



(More examples: Sipahi et al. [2].)

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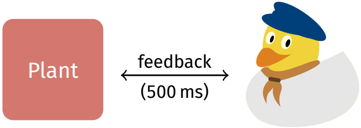

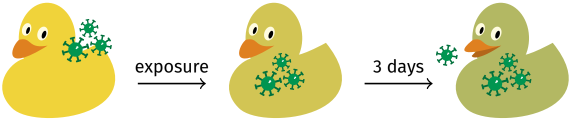

(More examples: Sipahi et al. [2].)

Delay differential systems

RDDE state space

$$\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau) + B\mathbf{u}(t),$$
$$\mathbf{y}(t) = C\mathbf{x}(t).$$

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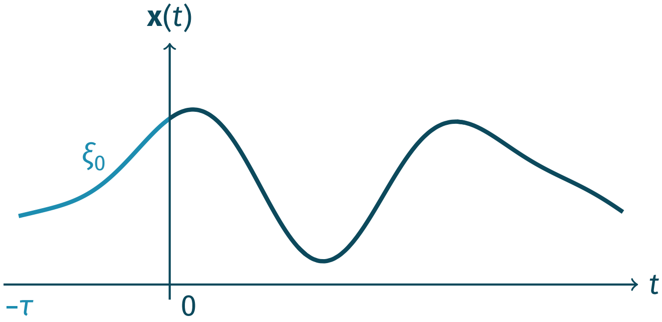
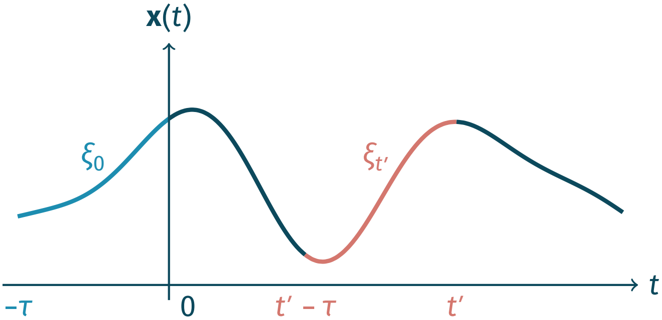
Transfer function

$$H(s) = C(sI_n - A_0 - A_1 e^{-\tau s})^{-1}B.$$

A functional differential equation



A functional differential equation



PDE formulation

$$\begin{cases} \dot{\xi}_t(0) = A_0 \xi_t(0) + A_1 \xi_t(-\tau) + B\mathbf{u}(t), \\ \dot{\xi}_t(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta} \xi_t(\theta), \end{cases}$$
$$\mathbf{y}(t) = C\xi_t(0),$$

where $\xi_t : [-\tau, 0] \to \mathbb{C}^n,$
$\theta \mapsto \mathbf{x}(t + \theta).$

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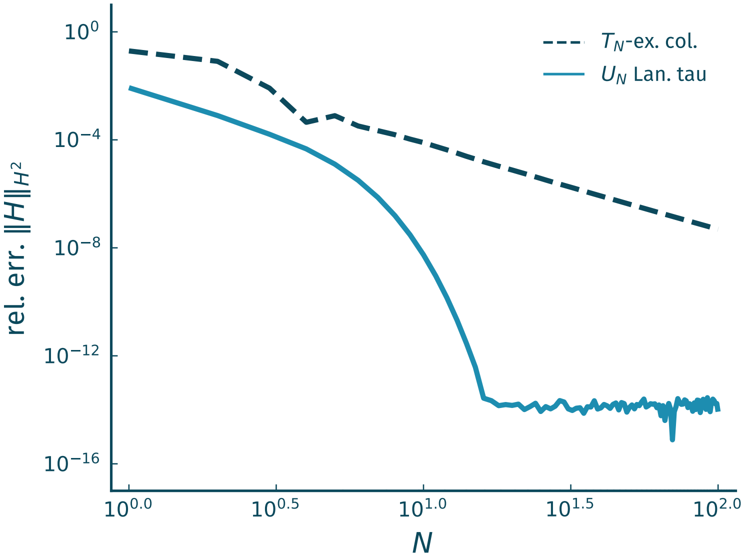
$$\text{where} \quad \xi_t : [-\tau, 0] \to \mathbb{C}^n, \qquad \theta \mapsto \mathbf{x}(t+\theta).$$

(See Breda et al. [3] for collocation.)

Why Lanczos tau?



Lanczos tau discretization

$$\begin{pmatrix} \varepsilon_0 \\ I \end{pmatrix} \dot{\xi}_t = \begin{pmatrix} A_0\varepsilon_0 + A_1\varepsilon_{-\tau} \\ \mathcal{D} \end{pmatrix} \xi_t + \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix} \mathbf{u}(t),$$

$$\mathbf{y}(t) = C\varepsilon_0\xi_t,$$

with $\varepsilon_\theta\xi = \xi(\theta)$ and $(\mathcal{D}\xi)(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}\xi(\theta)$.

Lanczos tau discretization

$$\begin{pmatrix} \mathcal{E}_0 \\ \mathcal{T}_{N-1} \end{pmatrix} \dot{\xi}_{tN} = \begin{pmatrix} A_0\mathcal{E}_0 + A_1\mathcal{E}_{-\tau} \\ \mathcal{D} \end{pmatrix} \xi_{tN} + \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix} \mathbf{u}(t),$$

$$\mathbf{y}(t) \approx C\mathcal{E}_0\xi_{tN},$$

with $\mathcal{T}_{N-1}\xi = \xi - \langle \xi, \phi_N \rangle \phi_N$.

(Initially presented by Ito and Teglas [4].)

$$\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau) + B\mathbf{u}(t)$$
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$$\mathbf{y}(t) = C\mathbf{x}(t)$$

$$\xrightarrow{\xi_t(\theta) = \mathbf{x}(t+\theta)}$$

$$\mathcal{E}\ \dot{\xi}_t = \mathcal{A}\ \xi_t + \mathcal{B}\ \mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathcal{C}\ \xi_t$$

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$$\xi_{tN}(\theta) \approx \mathbf{x}(t+\theta) \rightsquigarrow$$

$$\mathcal{E}_N\dot{\xi}_{tN} = \mathcal{A}_N\xi_{tN} + \mathcal{B}_N\mathbf{u}(t)$$
$$\mathbf{y}_N(t) = \mathcal{C}_N\xi_{tN}$$

Interpretation in frequency domain

$$\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau) + B\mathbf{u}(t)$$
$$\mathbf{y}(t) = C\mathbf{x}(t)$$

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$\Downarrow$ Laplace

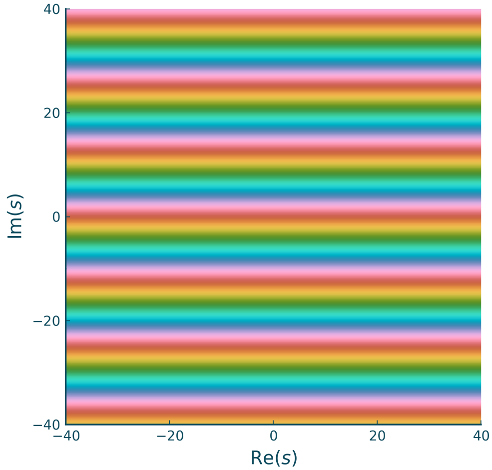
$$H(s) = C(sI_n - A_0 - A_1e^{-\tau s})^{-1}B$$

$\rightsquigarrow$

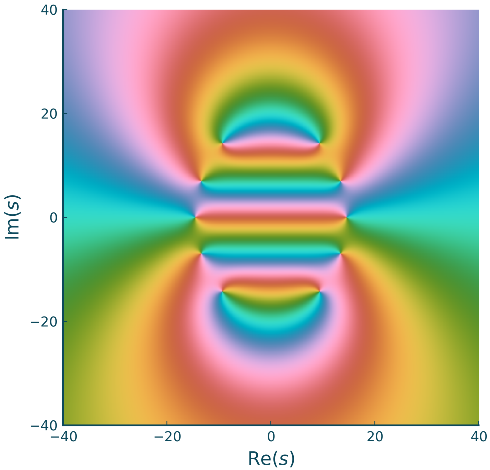
$$H_N(s) = C(sI_n - A_0 - A_1r_N(s,-\tau))^{-1}B$$

(See Vanbiervliet et al. [5].)

In the complex plane

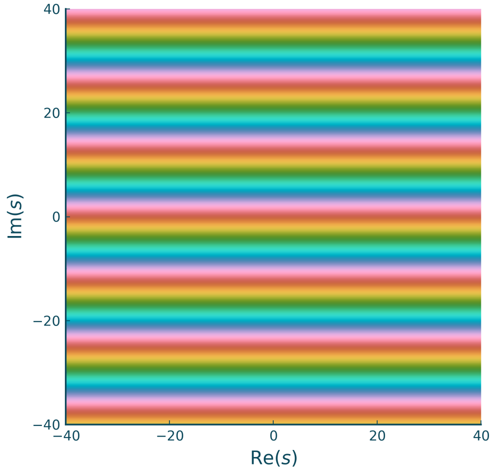


$s \mapsto e^{-\tau s}$ with $\tau = 1$



$s \mapsto r_N(s, -\tau)$ with $\phi_k = P_k$, $\tau = 1$, and $N = 5$

In the complex plane



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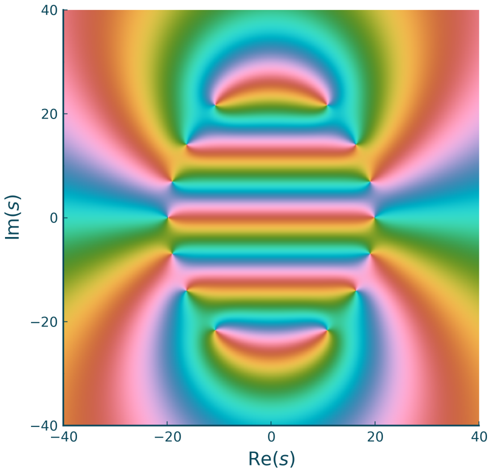

$s \mapsto r_N(s, -\tau)$ with $\phi_k = P_k$, $\tau = 1$, and $N = 7$

Padé approximation

Theorem

For the choice of $\{P_k\}_{k=0}^{N}$ as basis, $r_N(s, -\tau)$ is an (N, N) Padé approximant of $s \mapsto e^{-\tau s}$ around zero.

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Corollary

Then H_N is H with $s \mapsto e^{-\tau s}$ replaced by an (N, N) Padé approximant around zero.

Sparse, self-nesting discretizations

Clearly $\operatorname{span}\{\phi_k\}_{k=0}^{N_1} \subset \operatorname{span}\{\phi_k\}_{k=0}^{N_2}$ for $N_1 < N_2$, hence easy self-nesting.

(See Jarlebring et al. [6] and Olver and Townsend [7].)

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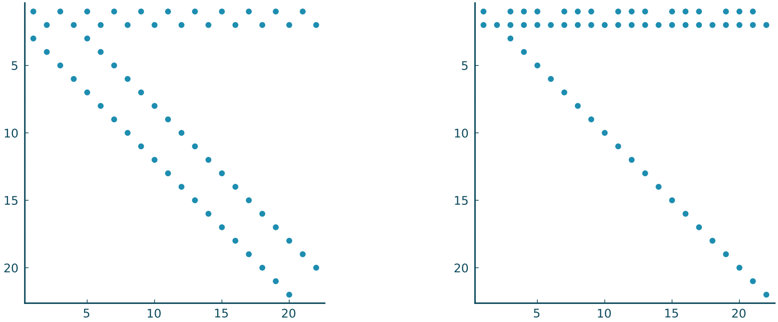


Simple example of an $(\mathcal{E}_N, \mathcal{A}_N)$ pencil at $n(N+1) = 22$.

(See Jarlebring et al. [6] and Olver and Townsend [7].)

What is the energy of
the impulse response?

An application: the H^2-norm

What is the energy of
the impulse response?

What is the steady-state power of
the output response to unit white noise?

The H^2-norm

Definition (stable system)

$$\|H\|_{H^2} = \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} \left\|H(i\omega)\right\|_F^2 \,\mathrm{d}\omega\right)^{\frac{1}{2}},$$

where $H(s) = C(sI_n - A_0 - A_1 e^{-\tau s})^{-1}B$.

Computing the H^2-norm of an ODE

$$E\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t),$$
$$\mathbf{y}(t) = C\mathbf{x}(t),$$

with E invertible.

$$\|H\|_{H^2} = \sqrt{\operatorname{tr}(CVC^T)}, \text{ where}$$

$$AVE^T + EVA^T = -BB^T.$$

(See Zhou et al. [8, Lemma 4.6].)

Approximating the H^2-norm of a RDDE

$$
\begin{aligned}
\mathcal{E}_N \dot{\mathbf{x}}(t) &= \mathcal{A}_N \mathbf{x}(t) + \mathcal{B}_N \mathbf{u}(t),\\
\mathbf{y}(t) &\approx \mathcal{C}_N \mathbf{x}(t),
\end{aligned}
$$

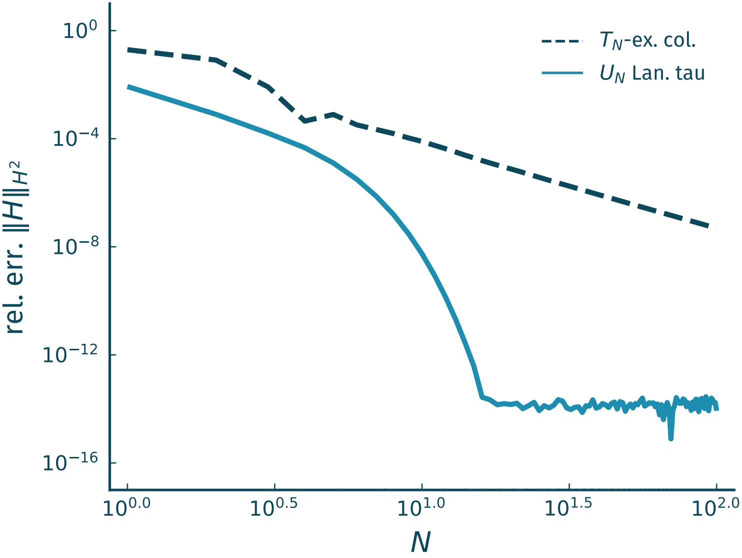
with $\mathcal{E}_N$ invertible.

$$\|H\|_{H^2} \approx \sqrt{\operatorname{tr}\left(\mathcal{C}_N V \mathcal{C}_N^T\right)}, \text{ where}$$

$$\mathcal{A}_N V \mathcal{E}_N^T + \mathcal{E}_N V \mathcal{A}_N^T = -\mathcal{B}_N \mathcal{B}_N^T.$$

(See Vanbiervliet et al. [5].)

Convergence



Polynomial formulation

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$$U(\theta, \theta') = \sum_{j,k} V_{jk}\, \phi_j(\theta)\phi_k(\theta') \in \mathbb{C}^{n \times n}.$$

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$\|H\|_{H^2} \approx \|H_N\|_{H^2} = \sqrt{\mathrm{tr}\big(\mathcal{C}_N V \mathcal{C}_N^T\big)}$, where

$$\mathcal{A}_N V \mathcal{E}_N^T + \mathcal{E}_N V \mathcal{A}_N^T = -\mathcal{B}_N \mathcal{B}_N^T.$$

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$\|H\|_{H^2} \approx \|H_N\|_{H^2} = \sqrt{\mathrm{tr}\left(C\varepsilon_0 U \varepsilon_0^T C^T\right)}$, where

$$\begin{cases} \mathcal{D}U\varepsilon_0^T + \mathcal{J}_{N-1}U\left(\varepsilon_0^T A_0^T + \varepsilon_{-\tau}^T A_1^T\right) = \mathbf{0}, \\ \mathcal{D}U\mathcal{J}_{N-1}^T + \mathcal{J}_{N-1}U\mathcal{D}^T = \mathbf{0}, \\ \varepsilon_\theta U \varepsilon_{\theta'}^T = \left(\varepsilon_{\theta'} U \varepsilon_\theta^T\right)^T, \\ (A_0\varepsilon_0 + A_1\varepsilon_{-\tau})U\varepsilon_0^T + \varepsilon_0 U\left(\varepsilon_0^T A_0^T + \varepsilon_{-\tau}^T A_1^T\right) = -BB^T. \end{cases}$$

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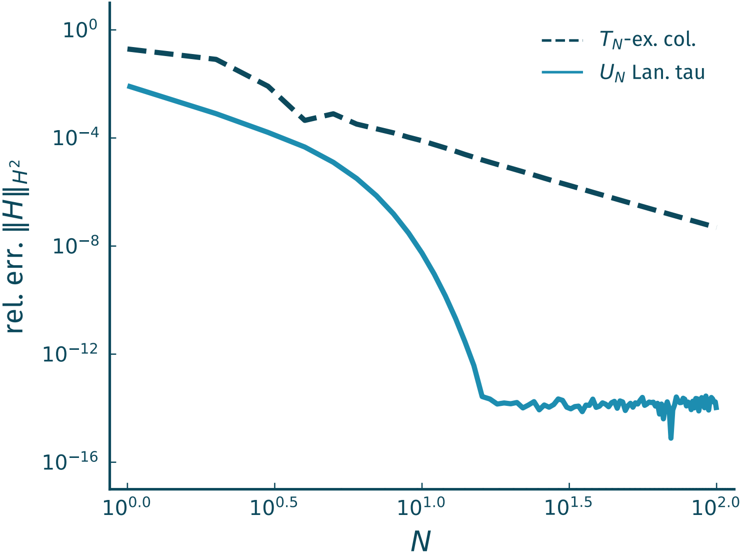
$\|H\|_{H^2} \approx \|H_N\|_{H^2} = \sqrt{\operatorname{tr}\left(C\varepsilon_0 U \varepsilon_0^T C^T\right)}$, where

some constraints on U.

Symmetry is important



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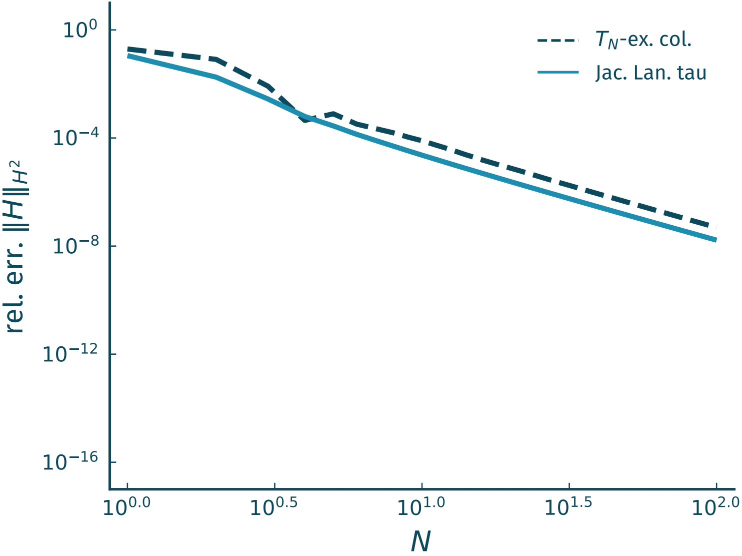

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Assumption

$$\phi_k(-\tau - \theta) = (-1)^k \phi_k(\theta), \quad \forall \theta \in [-\tau, 0].$$

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Proposition

Under this assumption

$$\left| r_N(i\omega, -\tau) \right| = 1, \quad \forall \omega \in \mathbb{R}.$$

Super convergence

Let $A_0 = A_1 = a < 0$,

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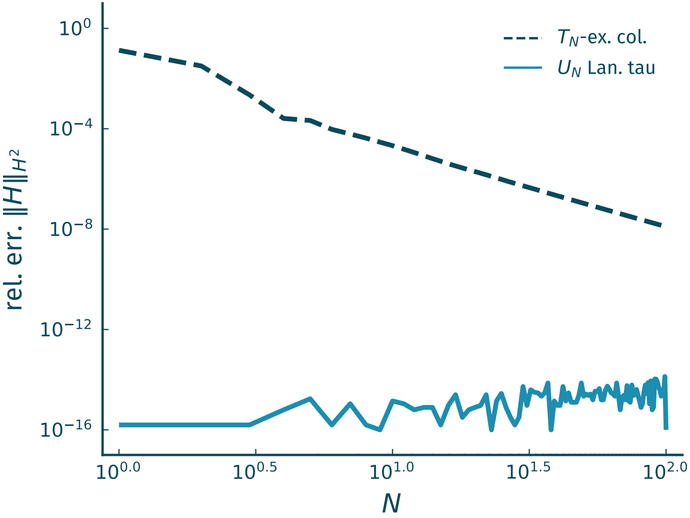

Super convergence

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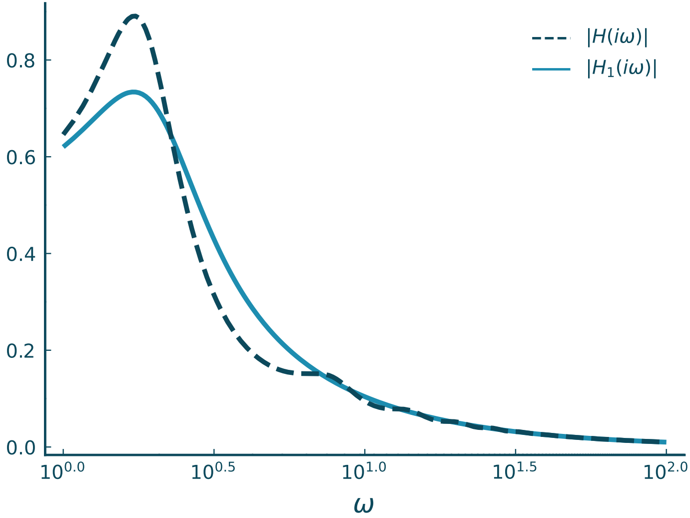

References

[1] E. Provoost and W. Michiels. *The Lanczos Tau Framework for Time-Delay Systems: Padé Approximation and Collocation Revisited*. 2024. arXiv: 2403.03895 [math.NA].

[2] R. Sipahi et al. “Stability and Stabilization of Systems with Time Delay”. In: *IEEE Control Systems* 31.1 (2011).

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[7] S. Olver and A. Townsend. “A Fast and Well-Conditioned Spectral Method”. In: *SIAM Review* 55.3 (2013).

[8] K. Zhou et al. *Robust and Optimal Control*. Pearson, 1995.

Contributions

Operator formulation of the Lanczos tau method for time-delay systems.

Equivalence to rational approximation in frequency domain, with explicit expressions.*

Construction of sparse, self-nesting discretizations.

Equivalence to pseudospectral collocation, with the non-zero collocation points the zeroes of ϕ_N.*

Equivalence to Padé approximation when using a Legendre basis.

Illustrated super-geometric convergence, and proved for some cases super convergence,* for the H^2-norm.

*Not in this talk, see E. Provoost and W. Michiels. *The Lanczos Tau Framework for Time-Delay Systems: Padé Approximation and Collocation Revisited*. 2024. arXiv: 2403.03895 [math.NA].