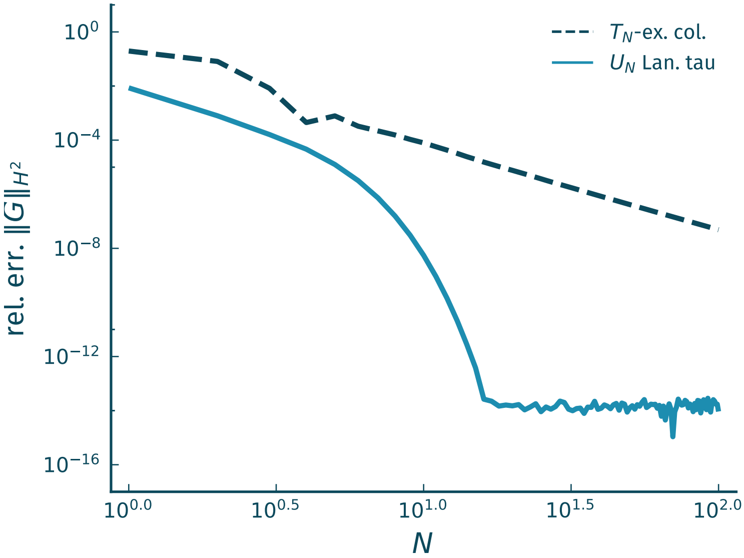
KU LEUVEN

# The Lanczos Tau Framework for Time-Delay Systems

Evert Provoost and Wim Michiels

# Why are we looking at Lanczos tau methods?



# In this talk: single delay RDDE

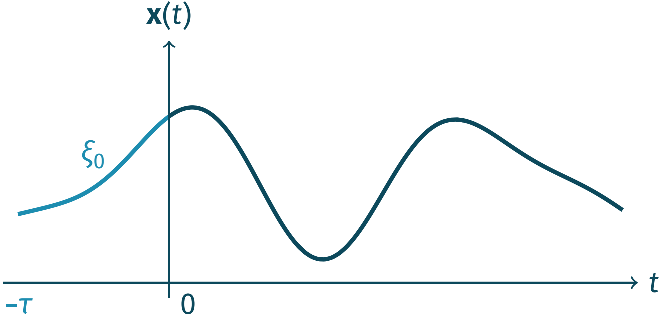
## State space

$$\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau) + B\mathbf{u}(t),$$
$$\mathbf{y}(t) = C\mathbf{x}(t).$$

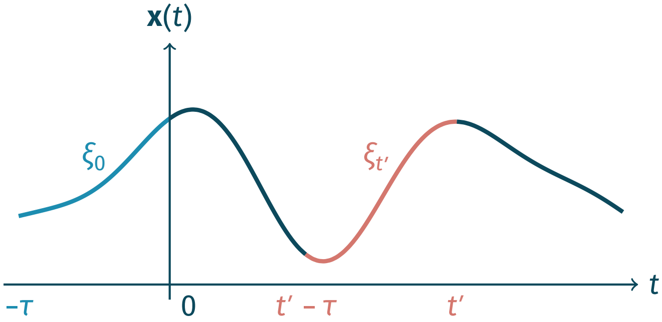
## Transfer function

$$G(s) = C(sI_n - A_0 - A_1 e^{-\tau s})^{-1}B.$$

# A functional differential equation



# A functional differential equation



# PDE formulation

$$\begin{cases} \dot{\xi}_t(0) = A_0\xi_t(0) + A_1\xi_t(-\tau) + B\mathbf{u}(t), \\ \dot{\xi}_t(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}\xi_t(\theta), \end{cases}$$
$$\mathbf{y}(t) = C\xi_t(0),$$

where $\xi_t : [-\tau, 0] \to \mathbb{C}^n,$
$\theta \mapsto \mathbf{x}(t + \theta).$

# PDE formulation

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$$\mathbf{y}(t) = C\xi_t(0).$$

Either collocate or *apply Lanczos tau method.*

(See Breda et al. [1] for collocation.)

# Lanczos tau discretization

$$\begin{pmatrix} \mathcal{E}_0 \\ I \end{pmatrix} \dot{\xi}_t = \begin{pmatrix} A_0\mathcal{E}_0 + A_1\mathcal{E}_{-\tau} \\ \mathcal{D} \end{pmatrix} \xi_t + \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix} \mathbf{u}(t),$$

$$\mathbf{y}(t) = C\mathcal{E}_0\xi_t,$$

with $\mathcal{E}_\theta\xi = \xi(\theta)$ and $(\mathcal{D}\xi)(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}\xi(\theta)$.

# Lanczos tau discretization

$$\begin{pmatrix} \varepsilon_0 \\ \mathcal{T}_{N-1} \end{pmatrix} \dot{\xi}_{tN} = \begin{pmatrix} A_0 \varepsilon_0 + A_1 \varepsilon_{-\tau} \\ \mathcal{D} \end{pmatrix} \xi_{tN} + \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix} \mathbf{u}(t),$$

$$\mathbf{y}_N(t) = C \varepsilon_0 \xi_{tN},$$

with $\mathcal{T}_{N-1}\xi = \xi - \langle \xi, \phi_N \rangle \phi_N.$

(Initially presented by Ito and Teglas [2].)

# Lanczos tau discretization

$$\begin{pmatrix} \mathcal{E}_0 \\ \mathcal{T}_{N-1} \end{pmatrix} \dot{\xi}_{tN} = \begin{pmatrix} A_0\mathcal{E}_0 + A_1\mathcal{E}_{-\tau} \\ \mathcal{D} \end{pmatrix} \xi_{tN} + \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix} \mathbf{u}(t),$$

$$\mathbf{y}_N(t) = C\mathcal{E}_0\xi_{tN},$$

with $\quad \mathcal{T}_{N-1}\xi = \xi - \langle \xi, \phi_N \rangle \phi_N.$

Note, $\begin{pmatrix} \mathcal{E}_0 \\ \mathcal{T}_{N-1} \end{pmatrix}$ is always invertible.

# Equivalence to pseudospectral collocation

$$\begin{pmatrix} \varepsilon_0 \\ \mathcal{J}_{N-1} \end{pmatrix} \dot{\xi}_{tN} = \begin{pmatrix} A_0 \varepsilon_0 + A_1 \varepsilon_{-\tau} \\ \mathcal{D} \end{pmatrix} \xi_{tN} + \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix} \mathbf{u}(t)$$

# Equivalence to pseudospectral collocation

$$\begin{cases} \dot{\xi}_{tN}(0) = A_0 \xi_{tN}(0) + A_1 \xi_{tN}(-\tau) + B\mathbf{u}(t), \\ \dot{\xi}_{tN} - \langle \dot{\xi}_{tN}, \phi_N \rangle \phi_N = \mathcal{D} \xi_{tN}. \end{cases}$$

# Equivalence to pseudospectral collocation

$$\begin{cases} \dot{\xi}_{tN}(0) = A_0 \xi_{tN}(0) + A_1 \xi_{tN}(-\tau) + B\mathbf{u}(t), \\ \dot{\xi}_{tN}(\theta_k) = \mathcal{D}\xi_{tN}(\theta_k) \quad \forall \phi_N(\theta_k) = 0. \end{cases}$$

# Equivalence to pseudospectral collocation

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**Theorem**

*A Lanczos tau method truncating in $\phi_N$ is equivalent to pseudospectral collocation in $\{0\} \cup \{\theta \in [-\tau, 0] : \phi_N(\theta) = 0\}$.*

$$\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t - \tau) + B\mathbf{u}(t)$$
$$\mathbf{y}(t) = C\mathbf{x}(t)$$

$$\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau) + B\mathbf{u}(t)$$
$$\mathbf{y}(t) = C\mathbf{x}(t)$$

$$\xrightarrow{\xi_t(\theta) = \mathbf{x}(t+\theta)}$$

$$\mathcal{E}\ \dot{\xi}_t = \mathcal{A}\ \xi_t + \mathcal{B}\ \mathbf{u}(t)$$
$$\mathbf{y}\ (t) = \mathcal{C}\ \xi_t$$

$$\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau) + B\mathbf{u}(t)$$
$$\mathbf{y}(t) = C\mathbf{x}(t)$$

$$\xi_{tN}(\theta) \approx \mathbf{x}(t+\theta) \rightsquigarrow$$

$$\mathcal{E}_N \dot{\xi}_{tN} = \mathcal{A}_N \xi_{tN} + \mathcal{B}_N \mathbf{u}(t)$$
$$\mathbf{y}_N(t) = \mathcal{C}_N \xi_{tN}$$

# Interpretation in frequency domain

$$\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau) + B\mathbf{u}(t)$$
$$\mathbf{y}(t) = C\mathbf{x}(t)$$

$\xi_{tN}(\theta) \approx \mathbf{x}(t+\theta)$

$$\mathcal{E}_N\dot{\xi}_{tN} = \mathcal{A}_N\xi_{tN} + \mathcal{B}_N\mathbf{u}(t)$$
$$\mathbf{y}_N(t) = \mathcal{C}_N\xi_{tN}$$

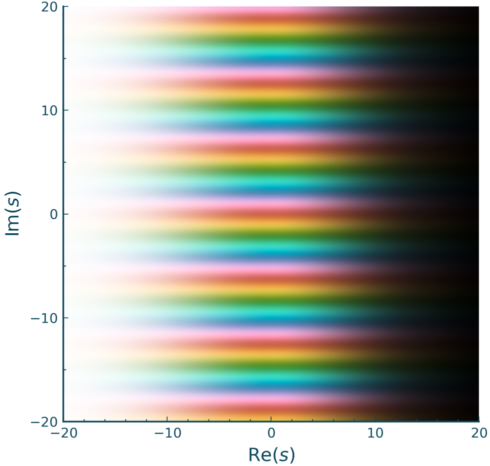
Laplace

$$G(s) = C(sI_n - A_0 - A_1 e^{-\tau s})^{-1}B$$

$$G_N(s) = C(sI_n - A_0 - A_1 r_N(s, -\tau))^{-1}B$$

(See Vanbiervliet et al. [3].)

# In the complex plane




$s \mapsto e^{-\tau s}$

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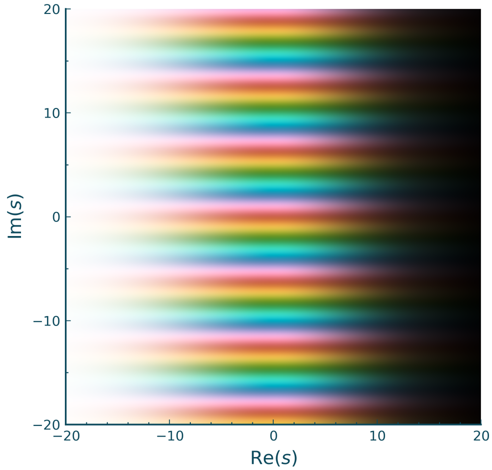


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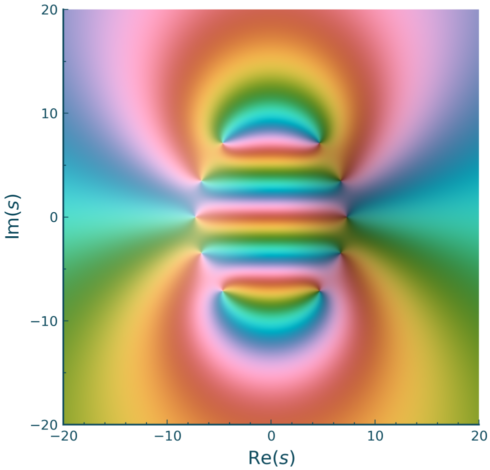
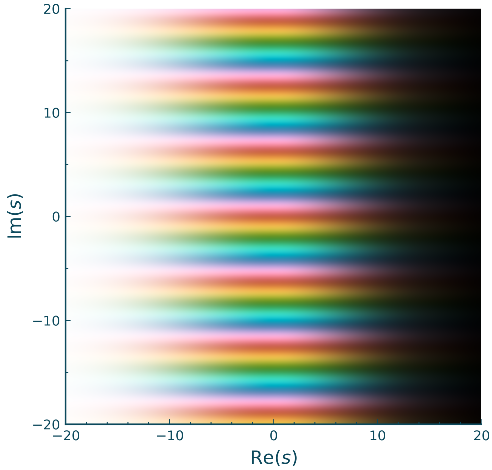



$s \mapsto r_N(s, -\tau)$ with $\phi_k = P_k$ and $N = 5$

# In the complex plane

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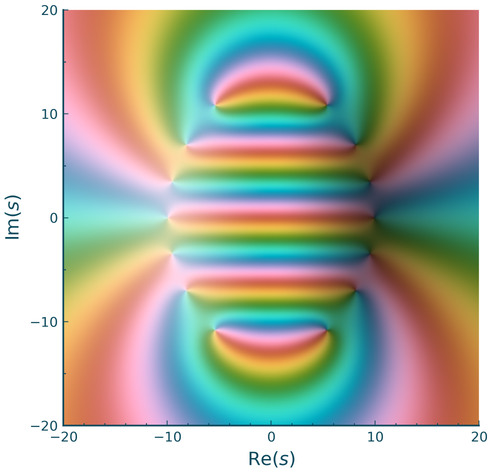


$s \mapsto r_N(s, -\tau)$ with $\phi_k = P_k$ and $N = 7$

**Theorem**

For the choice of $\{P_k\}_{k=0}^{N}$ as basis, $r_N(s, -\tau)$ is an $(N, N)$ Padé approximant of $s \mapsto e^{-\tau s}$ around zero.

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*Proof sketch*

$$\begin{cases} r_N(s,0) = 1, \\ \mathcal{D}r_N(s,\cdot) = s\mathcal{J}_{N-1}r_N(s,\cdot). \end{cases}$$

# Padé approximation

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For the choice of $\{P_k\}_{k=0}^N$ as basis, $r_N(s, -\tau)$ is an $(N, N)$ Padé approximant of $s \mapsto e^{-\tau s}$ around zero.

*Proof sketch*

$$r_N(s,\theta) = \varepsilon_\theta \begin{pmatrix} \varepsilon_0 \\ s\mathcal{T}_{N-1} - \mathcal{D} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} \implies \left[\frac{\mathrm{d}^n}{\mathrm{d}s^n} r_N(s,-\tau)\right]_{s=0} = n!\,\varepsilon_{-\tau}\mathcal{M}_N^n f_0,$$

where $(\mathcal{M}_N f)(\theta) = \int_0^\theta [f(\zeta) - \langle f, P_N\rangle P_N(\zeta)]\,\mathrm{d}\zeta$ and $f_0(\theta) = 1$.

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where $(\mathcal{M}_N f)(\theta) = \int_0^\theta [f(\zeta) - \langle f, P_N\rangle P_N(\zeta)]\,\mathrm{d}\zeta$ and $f_0(\theta) = 1$.

Then show $n!\,\varepsilon_{-\tau}\mathcal{M}_N^n f_0 = \left[\frac{\mathrm{d}^n}{\mathrm{d}s^n} e^{-\tau s}\right]_{s=0} = (-\tau)^n$, for $n = 0, \dots, 2N$.

# Padé approximation

**Theorem**

For the choice of $\{P_k\}_{k=0}^{N}$ as basis, $r_N(s, -\tau)$ is an $(N, N)$ Padé approximant of $s \mapsto e^{-\tau s}$ around zero.

**Corollary**

Then $G_N$ is $G$ with $s \mapsto e^{-\tau s}$ replaced by an $(N, N)$ Padé approximant around zero.

# Sparse, self-nesting discretizations

Clearly $\operatorname{span}\{\phi_k\}_{k=0}^{N_1} \subset \operatorname{span}\{\phi_k\}_{k=0}^{N_2}$ for $N_1 < N_2$, hence easy self-nesting.

(See Jarlebring et al. [4] and Olver and Townsend [5].)

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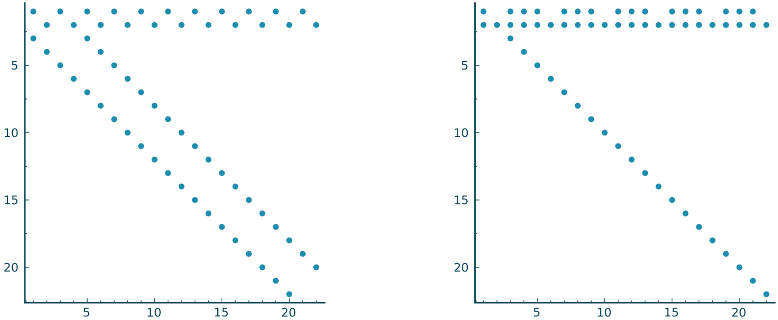



Simple example of an $(\mathcal{E}_N, \mathcal{A}_N)$ pencil at $n(N+1) = 22$.

(See Jarlebring et al. [4] and Olver and Townsend [5].)

# An application: the $H^2$-norm

**Definition (stable system)**

$$\|G\|_{H^2} = \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} \|G(i\omega)\|_F^2 \,\mathrm{d}\omega\right)^{\frac{1}{2}},$$

where $G(s) = C(sI_n - A_0 - A_1 e^{-\tau s})^{-1}B$.

# Computing the $H^2$-norm of an ODE

$$E\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t),$$
$$\mathbf{y}(t) = C\mathbf{x}(t),$$

with $E$ invertible.

$$\|G\|_{H^2} = \sqrt{\operatorname{tr}(CVC^T)}, \text{ where}$$

$$AVE^T + EVA^T = -BB^T.$$

(See Zhou et al. [6, Lemma 4.6].)

# Approximating the $H^2$-norm of a RDDE

$$\begin{aligned}\mathcal{E}_N\dot{\mathbf{x}}(t) &= \mathcal{A}_N\mathbf{x}(t) + \mathcal{B}_N\mathbf{u}(t),\\ \mathbf{y}(t) &\approx \mathcal{C}_N\mathbf{x}(t),\end{aligned}$$

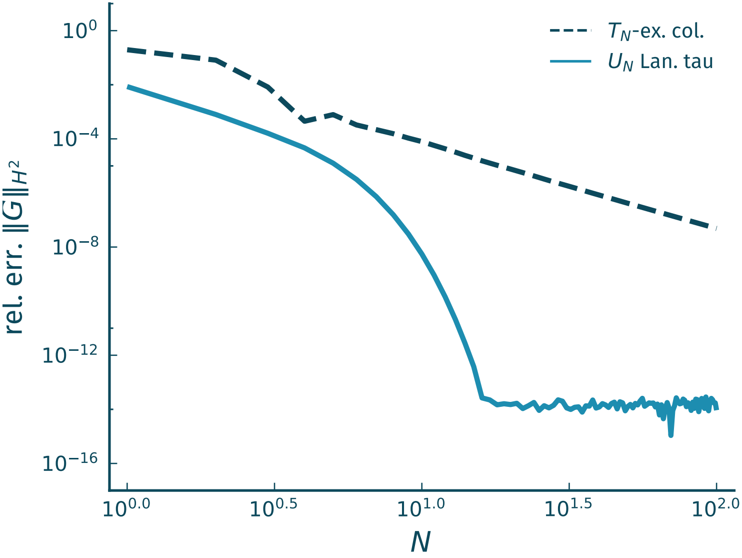
with $\mathcal{E}_N$ invertible.

$\|G\|_{H^2} \approx \sqrt{\operatorname{tr}\left(\mathcal{C}_N V \mathcal{C}_N^T\right)}$, where

$$\mathcal{A}_N V \mathcal{E}_N^T + \mathcal{E}_N V \mathcal{A}_N^T = -\mathcal{B}_N\mathcal{B}_N^T.$$

(See Vanbiervliet et al. [3].)

# Convergence



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$$U(\theta, \theta') = \sum_{j,k} V_{jk}\, \phi_j(\theta)\phi_k(\theta') \in \mathbb{C}^{n\times n}.$$

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$\|G\|_{H^2} \approx \|G_N\|_{H^2} = \sqrt{\mathrm{tr}\left(\mathcal{C}_N V \mathcal{C}_N^T\right)}$, where

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$\|G\|_{H^2} \approx \|G_N\|_{H^2} = \sqrt{\operatorname{tr}\left(C\varepsilon_0 U \varepsilon_0^T C^T\right)}$, where

$$\begin{pmatrix} A_0\varepsilon_0 + A_1\varepsilon_{-\tau} \\ \mathcal{D} \end{pmatrix} U \begin{pmatrix} \varepsilon_0 \\ \mathcal{J}_{N-1} \end{pmatrix}^T + \begin{pmatrix} \varepsilon_0 \\ \mathcal{J}_{N-1} \end{pmatrix} U \begin{pmatrix} A_0\varepsilon_0 + A_1\varepsilon_{-\tau} \\ \mathcal{D} \end{pmatrix}^T = -\begin{pmatrix} B \\ \mathbf{0} \end{pmatrix}\begin{pmatrix} B \\ \mathbf{0} \end{pmatrix}^T.$$

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$\|G\|_{H^2} \approx \|G_N\|_{H^2} = \sqrt{\operatorname{tr}\left(C\varepsilon_0 U\varepsilon_0^T C^T\right)}$, where

$$\begin{cases} \mathcal{D}U\varepsilon_0^T + \mathcal{T}_{N-1}U\left(\varepsilon_0^T A_0^T + \varepsilon_{-\tau}^T A_1^T\right) = \mathbf{0}, \\ \mathcal{D}U\mathcal{T}_{N-1}^T + \mathcal{T}_{N-1}U\mathcal{D}^T = \mathbf{0}, \\ \varepsilon_\theta U\varepsilon_{\theta'}^T = \left(\varepsilon_{\theta'}U\varepsilon_\theta^T\right)^T, \\ (A_0\varepsilon_0 + A_1\varepsilon_{-\tau})U\varepsilon_0^T + \varepsilon_0 U\left(\varepsilon_0^T A_0^T + \varepsilon_{-\tau}^T A_1^T\right) = -BB^T. \end{cases}$$

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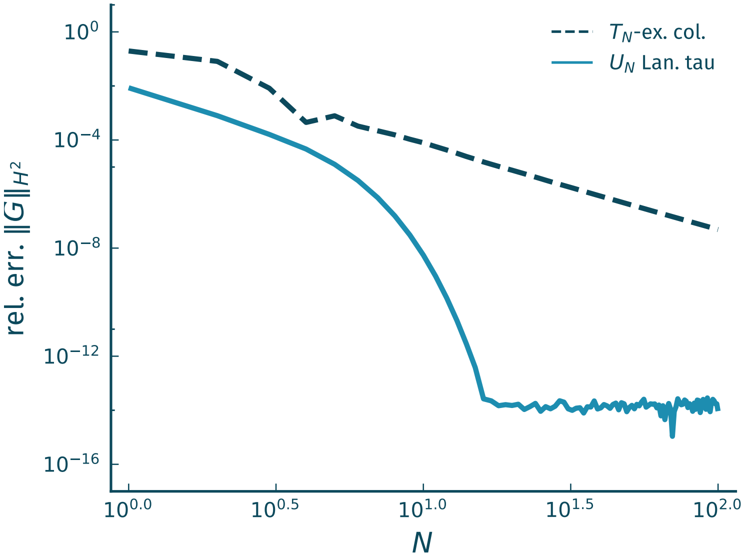
$$U(\theta, \theta') = \sum_{j,k} V_{jk}\, \phi_j(\theta)\phi_k(\theta') \in \mathbb{C}^{n\times n}.$$

$\|G\|_{H^2} \approx \|G_N\|_{H^2} = \sqrt{\mathrm{tr}\left(C\varepsilon_0 U \varepsilon_0^T C^T\right)}$, where

some constraints on $U$.

# Symmetry is important



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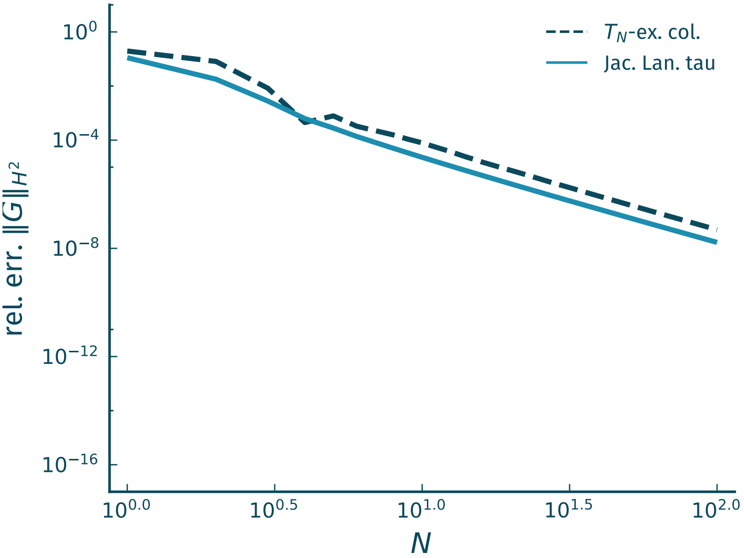

# Symmetry is important

## Assumption

$$\phi_k(-\tau - \theta) = (-1)^k \phi_k(\theta), \quad \forall \theta \in [-\tau, 0].$$

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$$\phi_k(-\tau - \theta) = (-1)^k \phi_k(\theta), \quad \forall \theta \in [-\tau, 0].$$

### Proposition

Under this assumption

$$\left| r_N(i\omega, -\tau) \right| = 1, \quad \forall \omega \in \mathbb{R}.$$

## Super convergence

Let $A_0 = A_1 = a < 0$,

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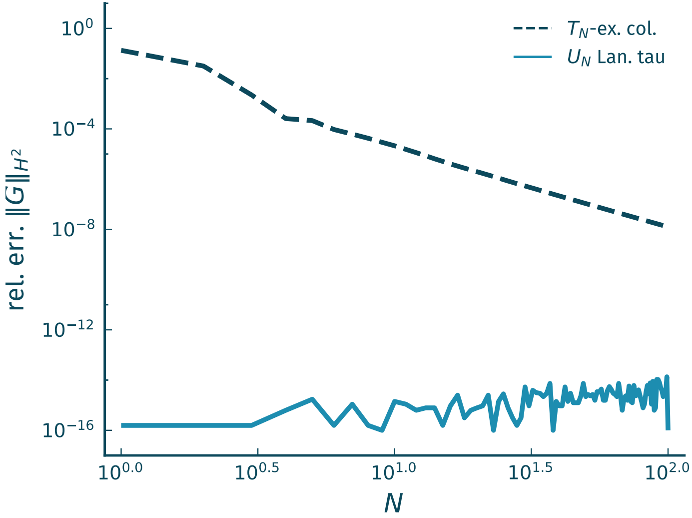

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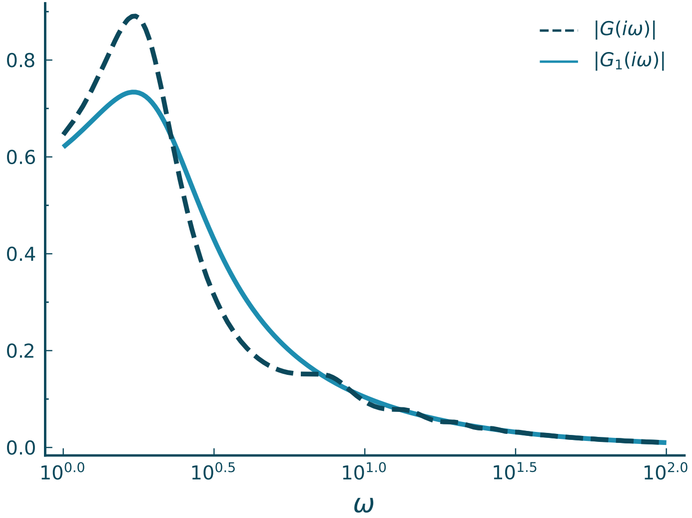

# References


[1] D. Breda et al. "Pseudospectral Differencing Methods for Characteristic Roots of Delay Differential Equations". In: *SIAM Journal on Scientific Computing* 27.2 (2005).

[2] K. Ito and R. Teglas. "Legendre–Tau Approximations for Functional-Differential Equations". In: *SIAM Journal on Control and Optimization* 24.4 (1986).

[3] J. Vanbiervliet et al. "Using spectral discretisation for the optimal $\mathcal{H}_2$ design of time-delay systems". In: *International Journal of Control* 84.2 (2011).

[4] E. Jarlebring et al. "A Krylov Method for the Delay Eigenvalue Problem". In: *SIAM Journal on Scientific Computing* 32.6 (2010).

[5] S. Olver and A. Townsend. "A Fast and Well-Conditioned Spectral Method". In: *SIAM Review* 55.3 (2013).

[6] K. Zhou et al. *Robust and Optimal Control*. Englewood Cliffs, NJ, 1995.

[7] EP and W. Michiels. "The Lanczos Tau Framework for Time-Delay Systems: Padé Approximation and Collocation Revisited". In: *SIAM Journal on Numerical Analysis* (accepted).

# Contributions

Operator formulation of the Lanczos tau method for time-delay systems.

Equivalence to rational approximation in frequency domain, with explicit expressions.

Construction of sparse, self-nesting discretizations.

Equivalence to pseudospectral collocation, with the non-zero collocation points the zeroes of $\phi_N$.

Equivalence to Padé approximation when using a Legendre basis.

Illustrated super-geometric convergence, and proved for some cases super convergence, for the $H^2$-norm.

For further details, see

# Explicit expression for the rational approximant

We have

$$\begin{cases} r_N(s,0) = 1, \\ \mathcal{D} r_N(s,\cdot) = s\mathcal{J}_{N-1} r_N(s,\cdot). \end{cases}$$

# Explicit expression for the rational approximant

We have

$$r_N(s,\theta) = \varepsilon_\theta \begin{pmatrix} \varepsilon_0 \\ s\mathcal{T}_{N-1} - \mathcal{D} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}.$$

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$$r_N(s,\theta) = \varepsilon_\theta \begin{pmatrix} \varepsilon_0 \\ s\mathcal{J}_{N-1} - \mathcal{D} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}.$$

In the derivative basis $\{\phi_N^{(N-k)}\}_{k=0}^N$, this becomes

$$r_N(s,\theta) = \begin{pmatrix} \phi_N^{(N)}(\theta) \\ \phi_N^{(N-1)}(\theta) \\ \vdots \\ \phi_N(\theta) \end{pmatrix}^T \begin{pmatrix} \phi_N^{(N)}(0) & \phi_N^{(N-1)}(0) & \cdots & \phi_N^{(1)}(0) & \phi_N(0) \\ s & -1 & & & \\ & s & -1 & & \\ & & \ddots & \ddots & \\ & & & s & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

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Thus, we get the explicit expression

$$r_N(s,\theta) = \frac{\sum_k \phi_N^{(N-k)}(\theta)\, s^k}{\sum_k \phi_N^{(N-k)}(0)\, s^k}.$$

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Analytically we find $\|G\|_{H^2}^2 = \frac{a\tau-1}{4a}$.

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For the approximation we have $\|G_N\|_{H^2}^2 = \varepsilon_0 U \varepsilon_0^T$, where

$$\begin{cases} \mathcal{D}U\varepsilon_0^T + a\mathcal{J}_{N-1}U\left(\varepsilon_0^T + \varepsilon_{-\tau}^T\right) = \mathbf{0}, \\ \mathcal{D}U\mathcal{J}_{N-1}^T + \mathcal{J}_{N-1}U\mathcal{D}^T = \mathbf{0}, \\ \varepsilon_\theta U \varepsilon_{\theta'}^T = \left(\varepsilon_{\theta'} U \varepsilon_\theta^T\right)^T, \\ a(\varepsilon_0 + \varepsilon_{-\tau})U\varepsilon_0^T + a\varepsilon_0 U\left(\varepsilon_0^T + \varepsilon_{-\tau}^T\right) = -1. \end{cases}$$

## Proving super convergence

Analytically we find $\|G\|_{H^2}^2 = \frac{a\tau - 1}{4a}$.

For the approximation we have $\|G_N\|_{H^2}^2 = \mu(0)$, where

$$\begin{cases} \mathcal{D}U\varepsilon_0^T + a\mathcal{J}_{N-1}U\left(\varepsilon_0^T + \varepsilon_{-\tau}^T\right) = \mathbf{0}, \\ \mathcal{D}U\mathcal{J}_{N-1}^T + \mathcal{J}_{N-1}U\mathcal{D}^T = \mathbf{0}, \\ \varepsilon_\theta U\varepsilon_{\theta'}^T = \left(\varepsilon_{\theta'}U\varepsilon_\theta^T\right)^T, \\ a(\varepsilon_0 + \varepsilon_{-\tau})U\varepsilon_0^T + a\varepsilon_0 U\left(\varepsilon_0^T + \varepsilon_{-\tau}^T\right) = -1. \end{cases}$$

Let $\mu = U\varepsilon_0^T$.

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Let $\mu = U\varepsilon_0^T$. From Lemma we have $U\varepsilon_{-\tau}^T = \mathcal{R}U\varepsilon_0^T = \mathcal{R}\mu$.

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For $N \geq 1$, this has the unique solution $\mu(\theta) = \frac{a\tau + 2a\theta - 1}{4a}$. □