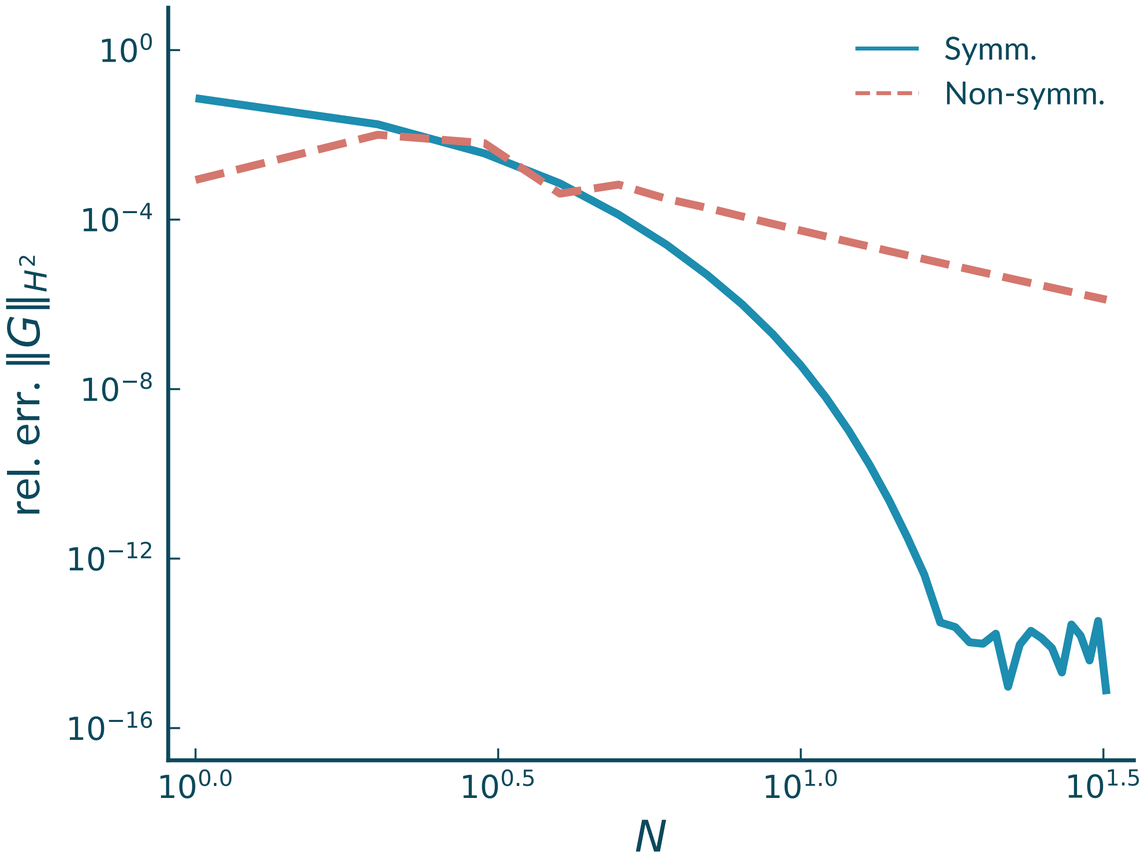
KU LEUVEN

Accelerated H^2-norm Approximation for Time-Delay Systems with Discrete Delays

Evert Provoost and Wim Michiels

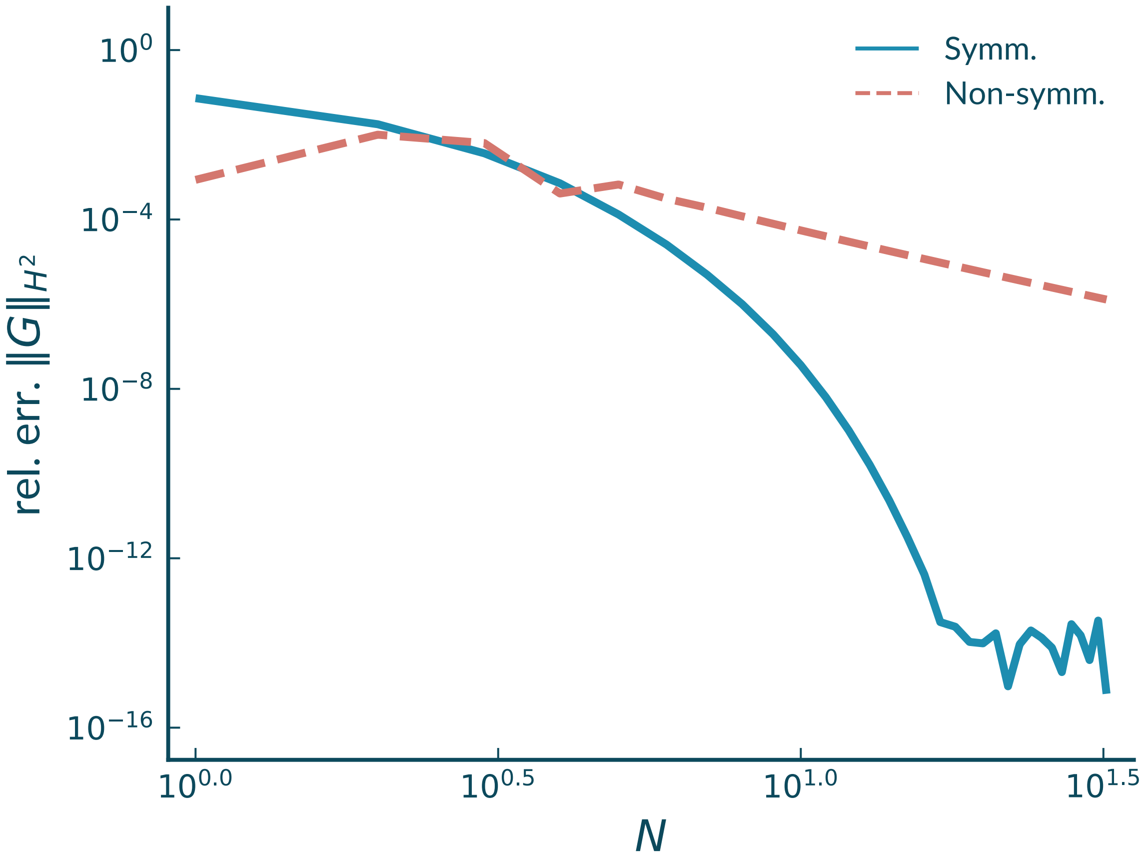
The backstory



$$m = 1$$

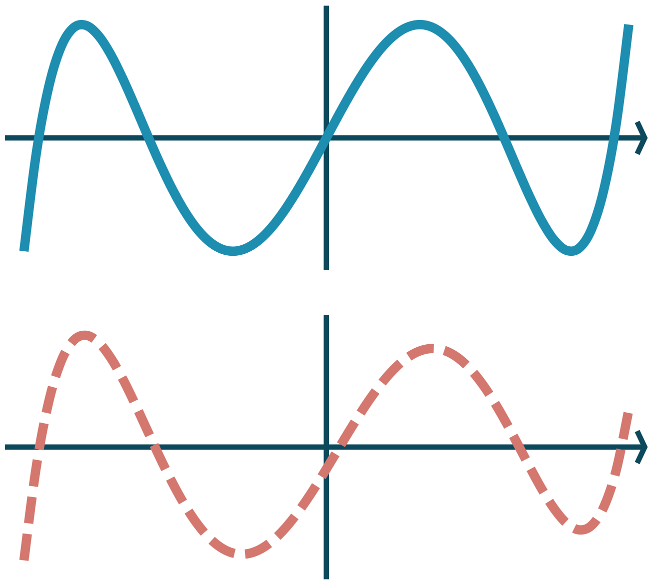
See Provoost and Michiels (2024).

The backstory



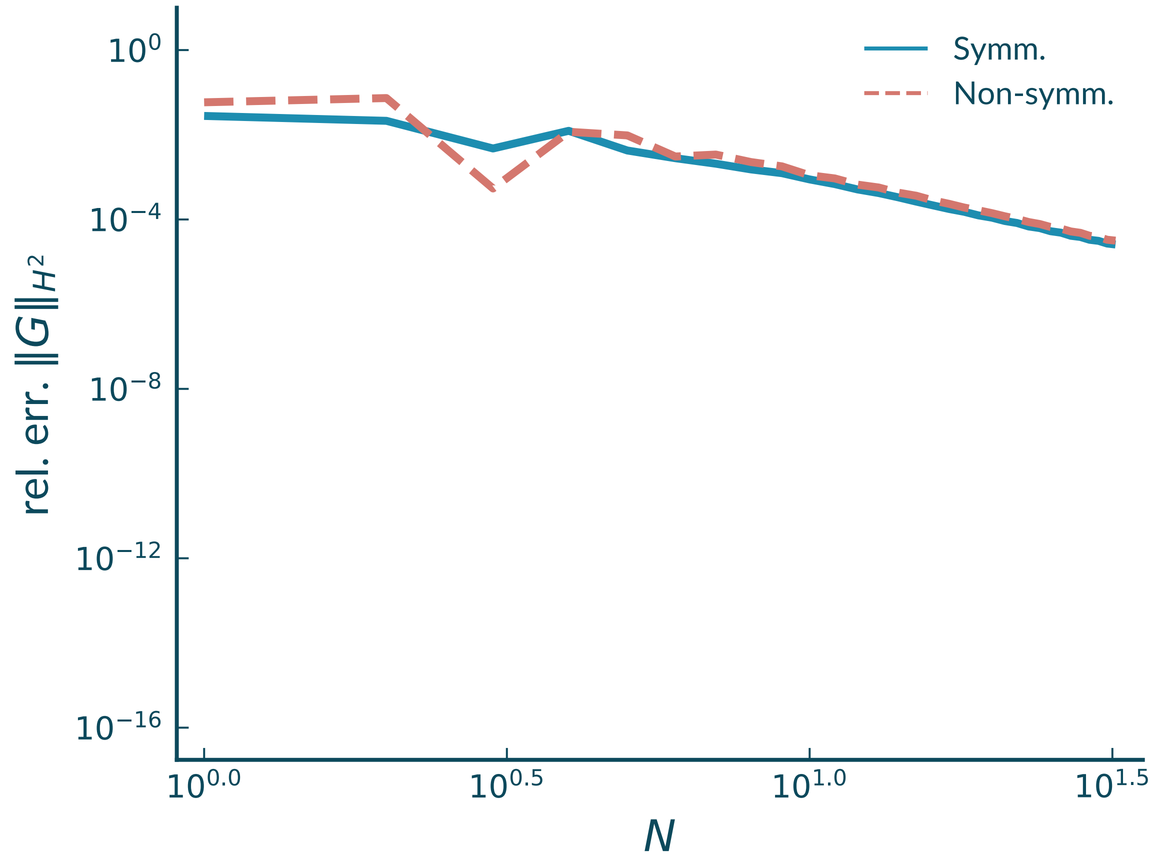
$m = 1$

Symmetry is required

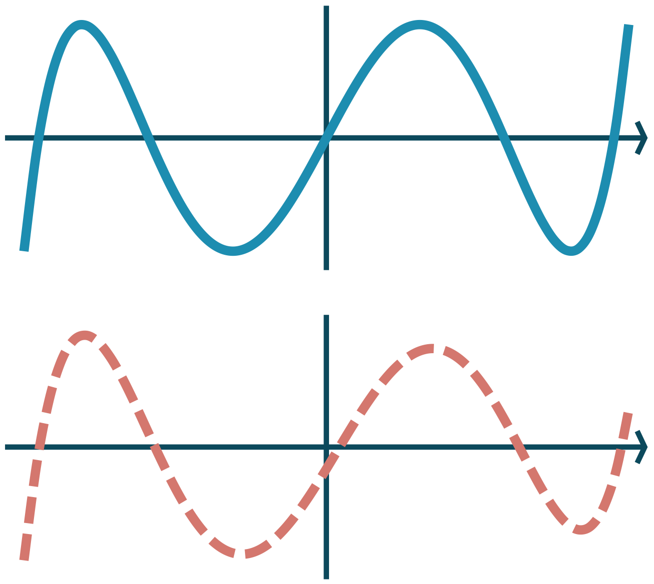


See Provoost and Michiels (2024).

The backstory



$m = 1$ is required



$m > 1$

See Provoost and Michiels (2024).

Our aim today

Approximate

$$\|G\|_{H^2} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \|G(i\omega)\|_F^2 \, \mathrm{d}\omega \right)^{\frac{1}{2}}$$

of an exponentially stable system.

Our aim today

Approximate

$$\|G\|_{H^2} = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \|G(i\omega)\|_F^2 \,\mathrm{d}\omega\right)^{\frac{1}{2}},$$

of an exponentially stable system

$$\dot{\mathbf{x}}(t) = \sum_{k=0}^{m} A_k \mathbf{x}(t-\tau_k) + B\mathbf{u}(t),$$
$$\mathbf{y}(t) = C\mathbf{x}(t),$$

where $\tau_0 = 0$.

Our aim today

Approximate

$$\|G\|_{H^2} = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \|G(i\omega)\|_F^2 \,\mathrm{d}\omega\right)^{\frac{1}{2}},$$

of an exponentially stable system

$$\dot{\mathbf{x}}(t) = \textstyle\sum_{k=0}^{m} A_k \mathbf{x}(t-\tau_k) + B\mathbf{u}(t),$$
$$\mathbf{y}(t) = C\mathbf{x}(t),$$

where $\tau_0 = 0$, and

$$G(s) = C\big(sI_n - \textstyle\sum_{k=0}^{m} A_k e^{-\tau_k s}\big)^{-1}B.$$

The delay-free case

If $m = 0$, we have

$$\|G\|_{H^2} = \sqrt{\operatorname{tr}(CVC^T)},$$

where V solves

$$A_0 V + V A_0^T = -BB^T.$$

The delay-free case → delay case

If $m = 0$, we have

$$\|G\|_{H^2} = \sqrt{\mathrm{tr}(CVC^T)},$$

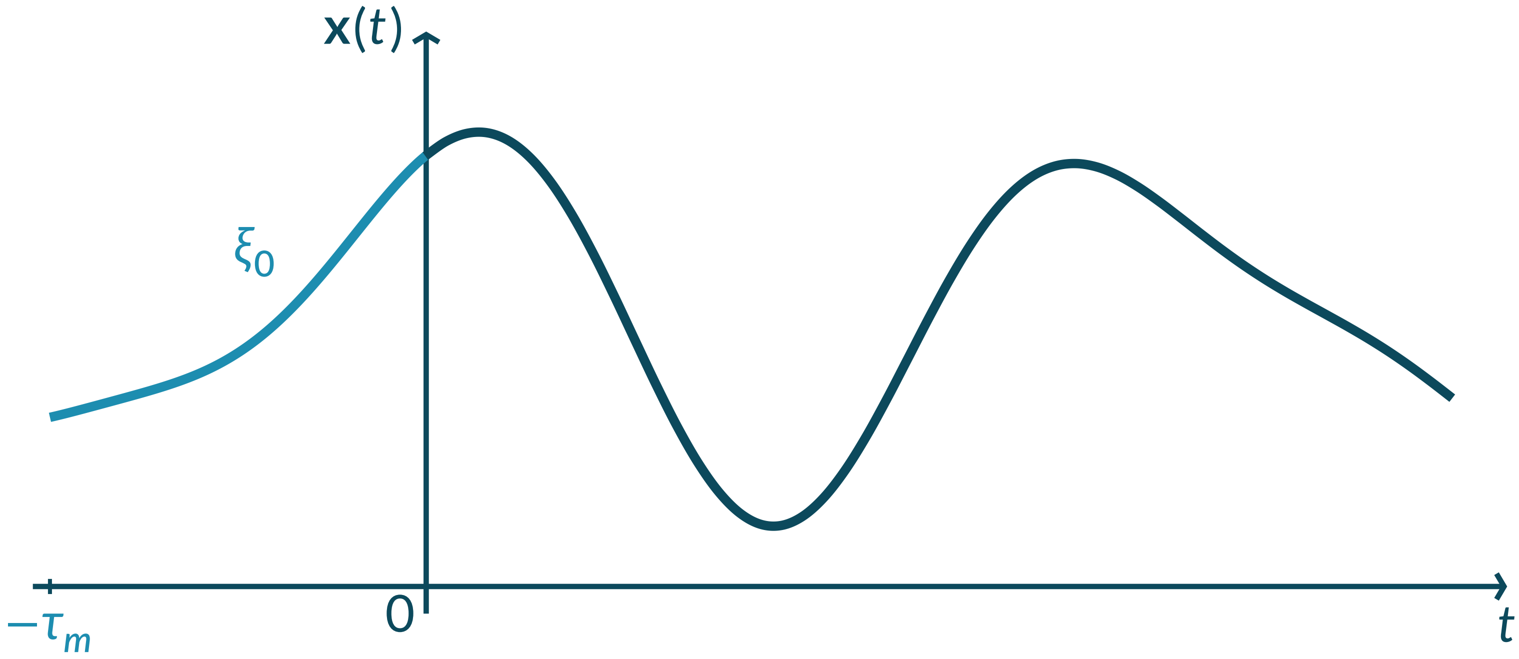
where V solves

$$A_0 V + V A_0^T = -BB^T.$$

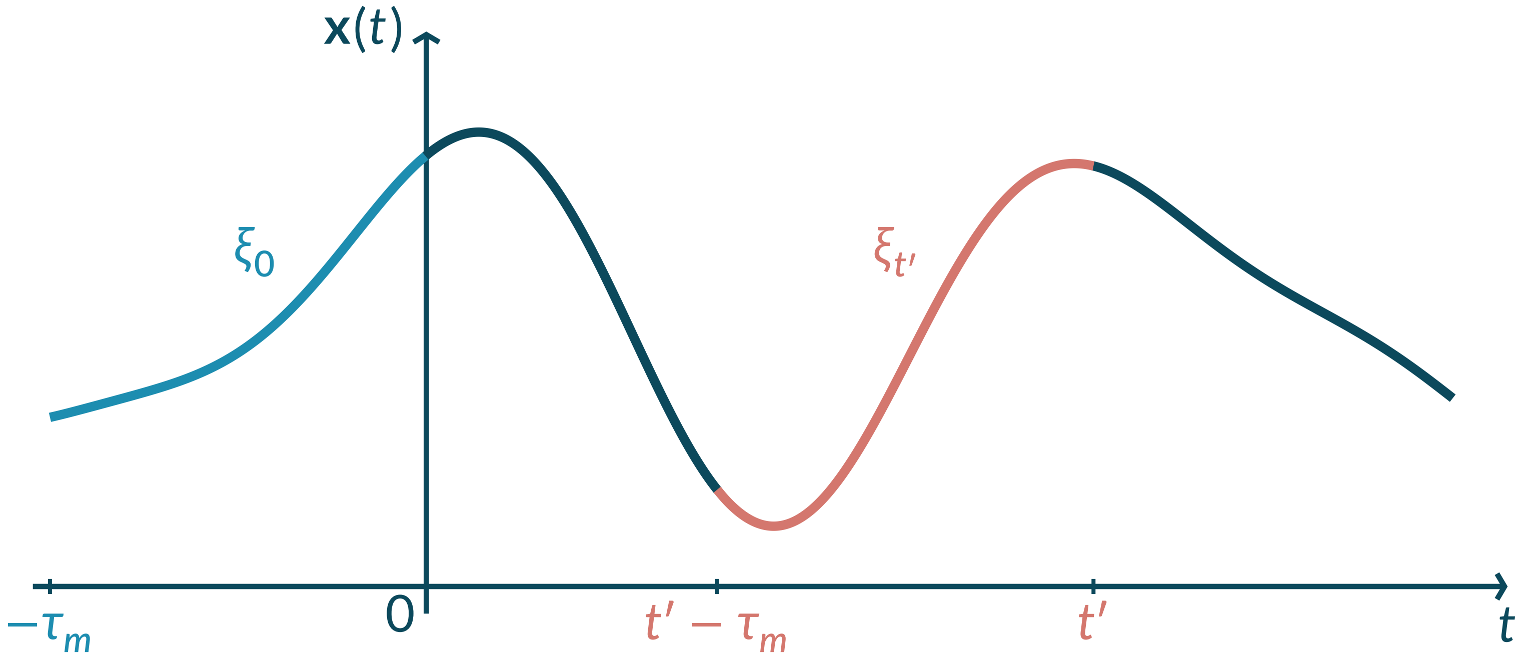
$m \neq 0 \implies$ use the H^2-norm of a delay-free approximation.

See Vanbiervliet, Michiels, and Jarlebring (2011).

Head-tail representation



Head-tail representation



Lanczos tau method

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= \sum_{k=0}^{m} A_k \mathbf{x}(t-\tau_k) + B\mathbf{u}(t),\\
\mathbf{y}(t) &= C\mathbf{x}(t).
\end{aligned}
$$

See Ito and Teglas (1986) and Provoost and Michiels (2024).

Lanczos tau method

$$\begin{cases} \dot{\xi}_t(0) = \sum_{k=0}^{m} A_k \xi_t(-\tau_k) + B\mathbf{u}(t), \\ \dot{\xi}_t(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}\xi_t(\theta), \end{cases}$$

$$\mathbf{y}(t) = C\xi_t(0),$$

where $\xi_t(\theta) = \mathbf{x}(t+\theta)$ with $\theta \in [-\tau_m, 0]$.

See Ito and Teglas (1986) and Provoost and Michiels (2024).

Lanczos tau method

$$\begin{pmatrix} \varepsilon_0 \\ \mathrm{Id} \end{pmatrix} \dot{\xi}_t = \begin{pmatrix} \sum_{k=0}^{m} A_k \varepsilon_{-\tau_k} \\ \mathcal{D} \end{pmatrix} \xi_t + \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix} \mathbf{u}(t),$$

$$\mathbf{y}(t) = C \varepsilon_0 \xi_t,$$

where $\varepsilon_\theta \xi = \xi(\theta)$ and $\mathcal{D}\xi(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}\xi(\theta)$.

See Ito and Teglas (1986) and Provoost and Michiels (2024).

Lanczos tau method

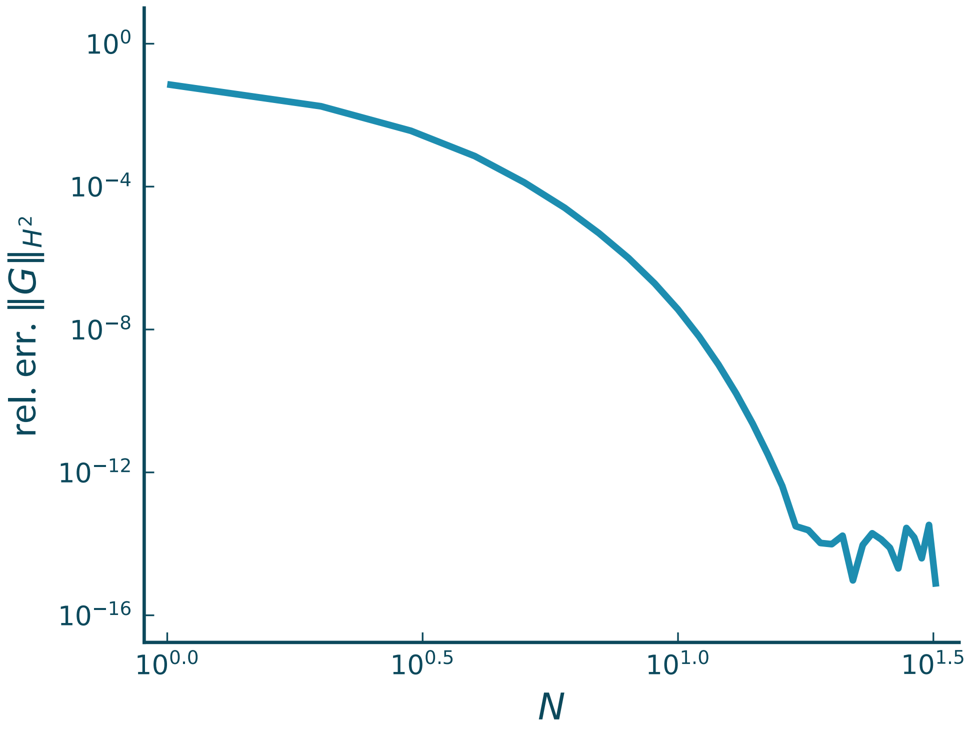
$$\begin{pmatrix} \varepsilon_0 \\ \mathcal{T}_{\varphi_N} \end{pmatrix} \dot{\xi}_{tN} = \begin{pmatrix} \sum_{k=0}^{m} A_k \varepsilon_{-\tau_k} \\ \mathcal{D} \end{pmatrix} \xi_{tN} + \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix} \mathbf{u}(t),$$

$$\mathbf{y}_N(t) = C \varepsilon_0 \xi_{tN},$$

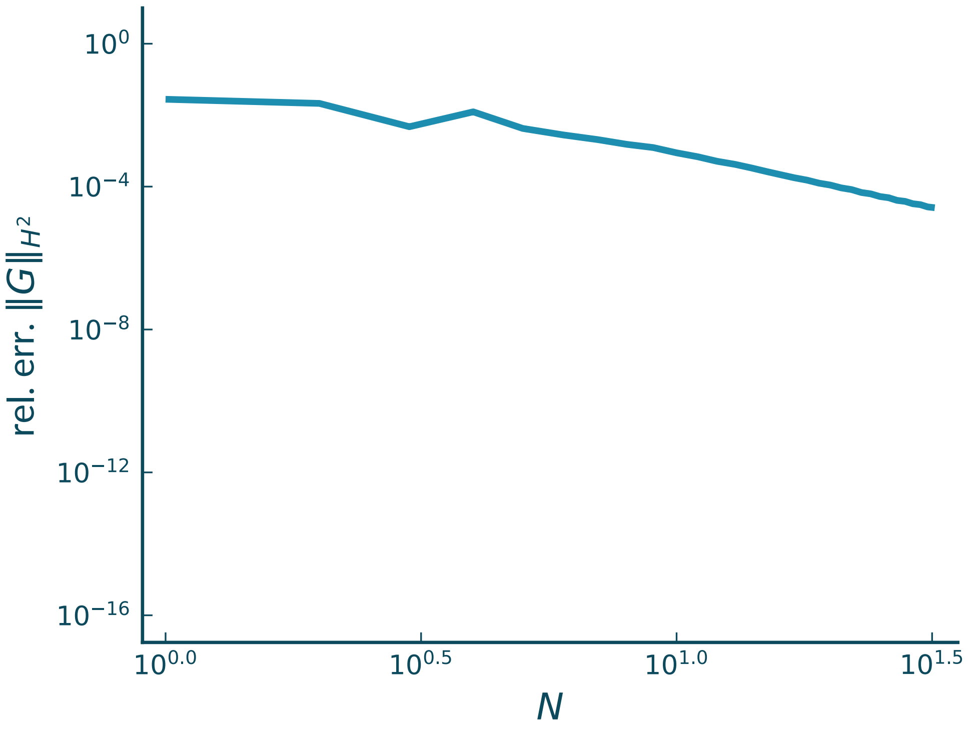
where $\mathcal{T}_{\varphi_N} \xi = \xi - \langle \xi, \varphi_N \rangle \frac{\varphi_N}{\|\varphi_N\|^2}$.

See Ito and Teglas (1986) and Provoost and Michiels (2024).

Convergence



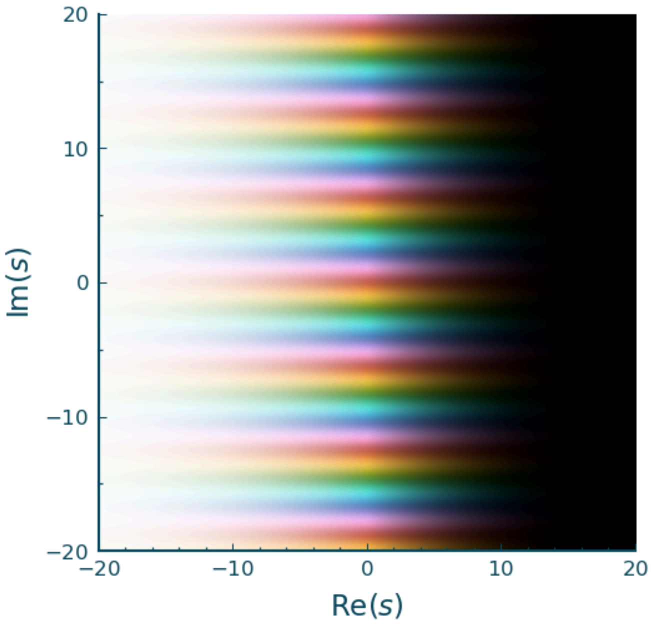
$m = 1$



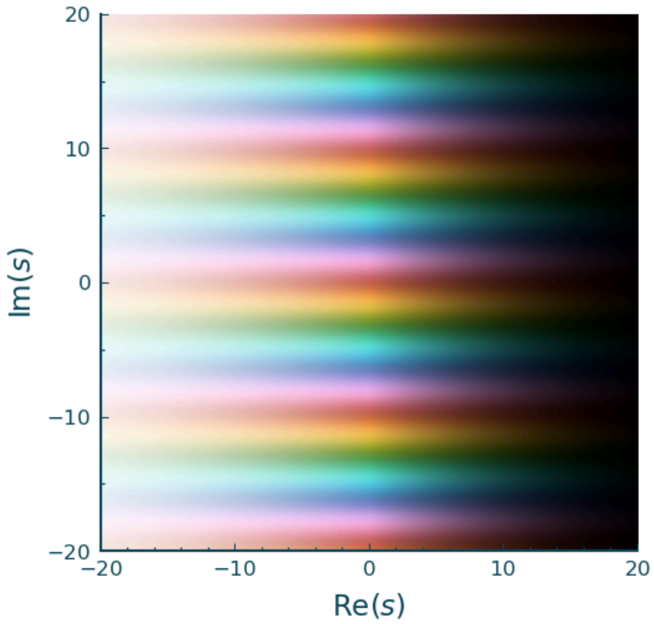
$m > 1$

In the frequency domain

$$G(s) = C\Big(sI_n - \sum_{k=0}^{m} A_k e^{-\tau_k s} \qquad \Big)^{-1} B$$



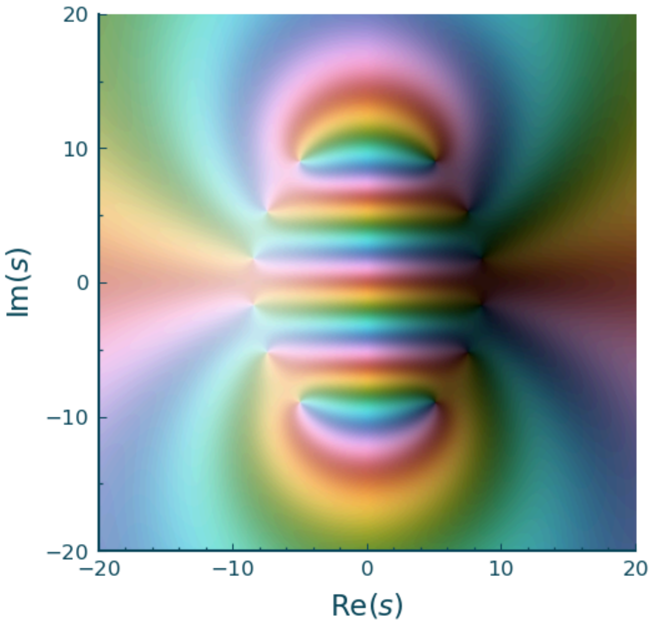
τ_m



τ_1

In the frequency domain

$$G_N(s) = C\left(sI_n - \sum_{k=0}^{m} A_k r_N(s, -\tau_k)\right)^{-1} B$$

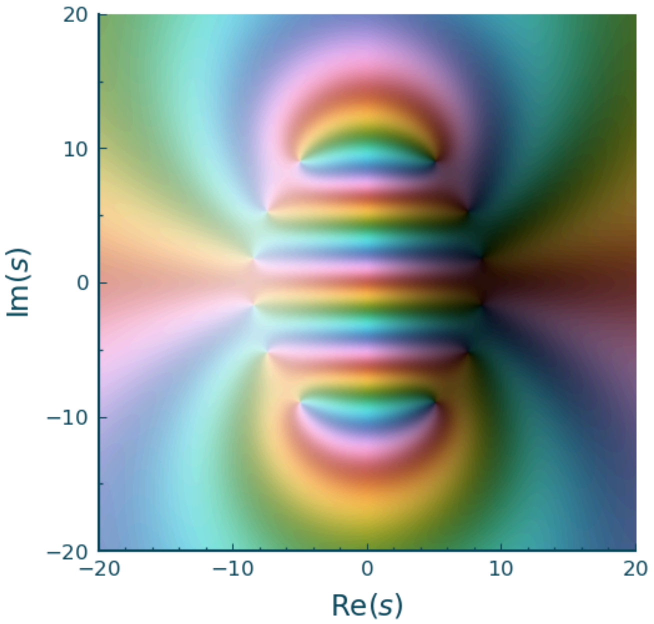


τ_m

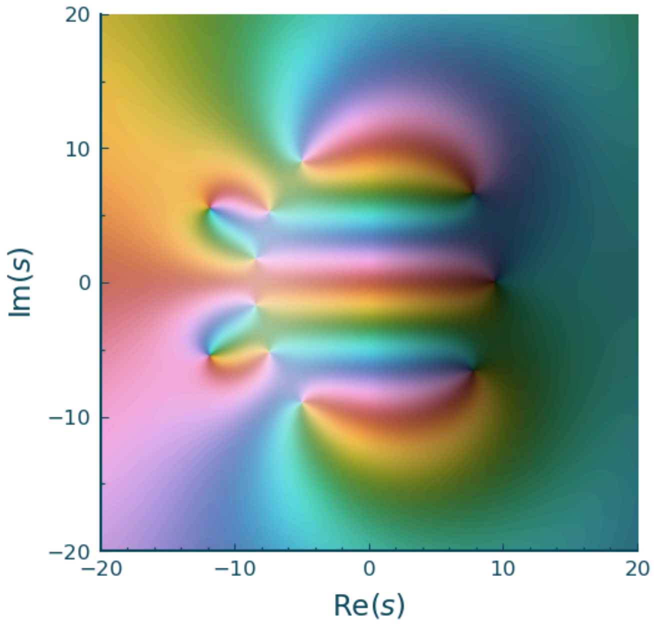
See Provoost and Michiels (2024).

In the frequency domain

$$G_N(s) = C\left(sI_n - \sum_{k=0}^{m} A_k r_N(s, -\tau_k)\right)^{-1} B$$



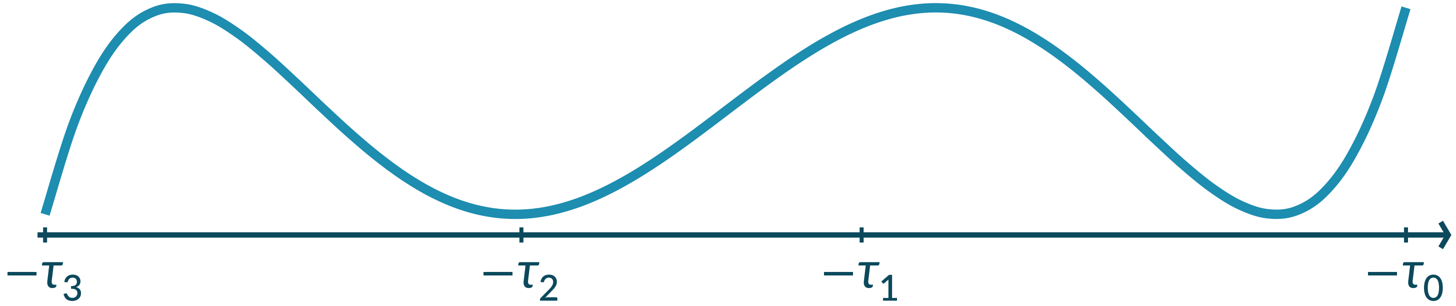
τ_m



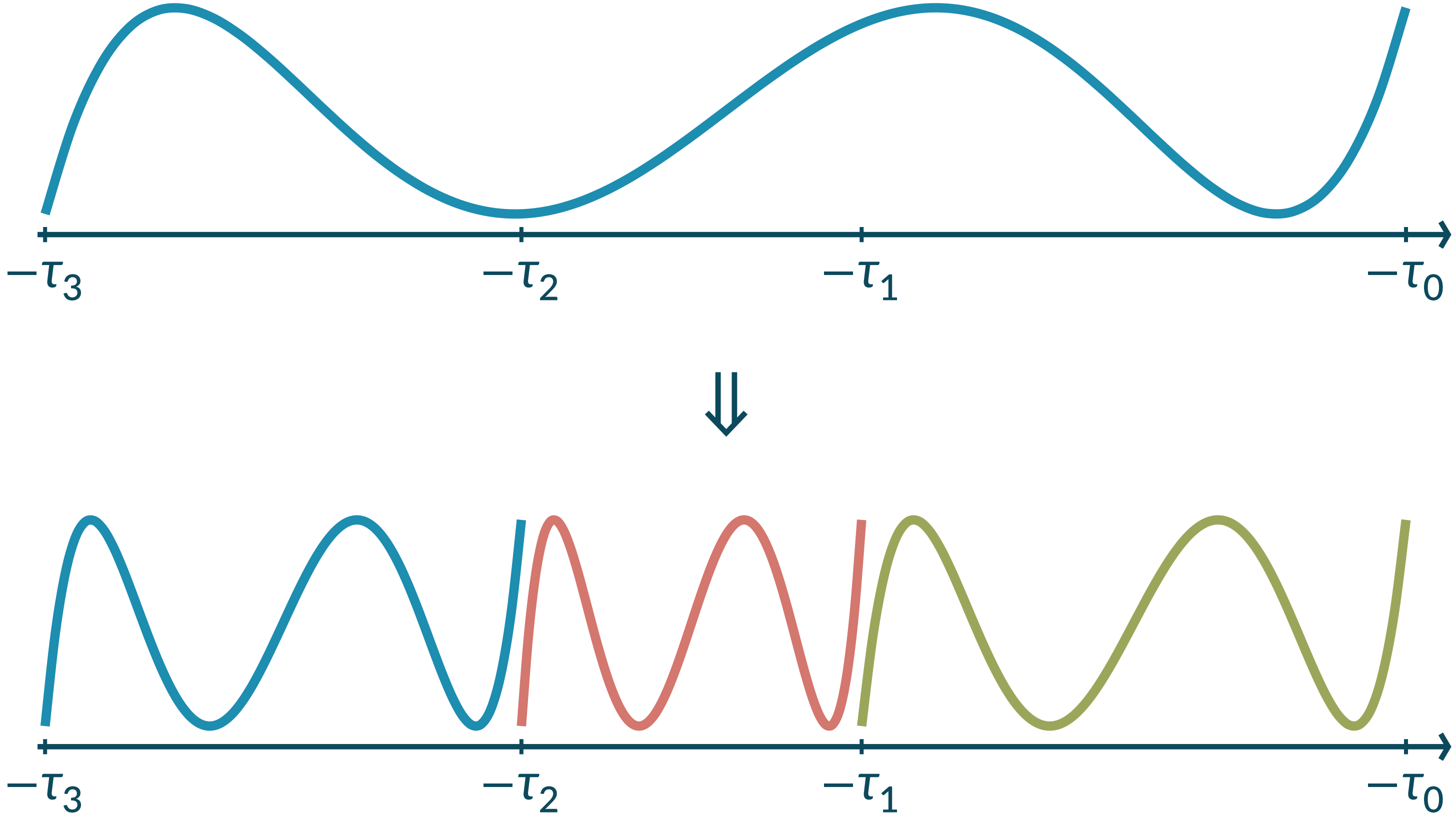
τ_1

See Provoost and Michiels (2024).

Splines



Splines



Spline Lanczos tau method

$$\begin{pmatrix} \varepsilon_0^{(1)} \\ \left(\mathcal{T}_{\varphi_{k,N}}^{(k)}\right)_{k=1}^{m} \\ \left(\varepsilon_{-\tau_k}^{(k)} - \varepsilon_{-\tau_k}^{(k+1)}\right)_{k=1}^{m-1} \end{pmatrix} \dot{\Xi}_{tN} = \begin{pmatrix} A_0 \varepsilon_0^{(1)} + \sum_{k=1}^{m} A_k \varepsilon_{-\tau_k}^{(k)} \\ \left(\mathcal{D}^{(k)}\right)_{k=1}^{m} \\ (\mathbf{0})_{k=1}^{m-1} \end{pmatrix} \Xi_{tN} + \begin{pmatrix} B \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \mathbf{u}(t),$$

$$\mathbf{y}_N(t) = C\varepsilon_0^{(1)} \Xi_{tN},$$

where $\Xi_{tN} = \{\xi_{tN}^{(k)} : [-\tau_k, -\tau_{k-1}] \to \mathbb{C}^n\}_{k=1}^{m}$ is a continuous spline.

See also Ito and Teglas (1987) and Breda, Maset, and Vermiglio (2005).

Spline Lanczos tau method

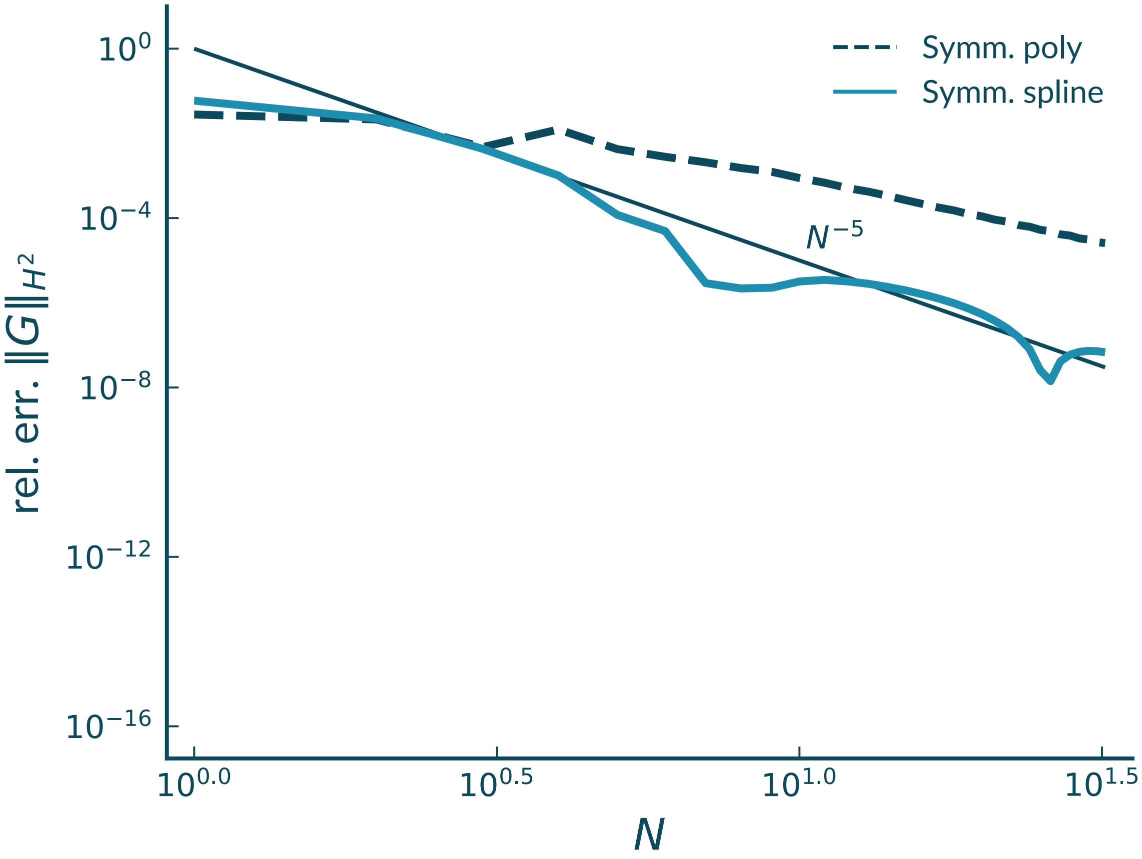
$$\begin{pmatrix} \varepsilon_0^{(1)} \\ \left(\mathcal{T}^{(k)}_{\varphi_{k,N}}\right)_{k=1}^{m} \\ \left(\varepsilon_{-\tau_k}^{(k)} - \varepsilon_{-\tau_k}^{(k+1)}\right)_{k=1}^{m-1} \end{pmatrix} \dot{\Xi}_{tN} = \begin{pmatrix} A_0 \varepsilon_0^{(1)} + \sum_{k=1}^{m} A_k \varepsilon_{-\tau_k}^{(k)} \\ \left(\mathcal{D}^{(k)}\right)_{k=1}^{m} \\ -\left(\varepsilon_{-\tau_k}^{(k)} - \varepsilon_{-\tau_k}^{(k+1)}\right)_{k=1}^{m-1} \end{pmatrix} \Xi_{tN} + \begin{pmatrix} B \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \mathbf{u}(t),$$

$$\mathbf{y}_N(t) = C\varepsilon_0^{(1)} \Xi_{tN},$$

where $\Xi_{tN} = \{\xi_{tN}^{(k)} : [-\tau_k, -\tau_{k-1}] \to \mathbb{C}^n\}_{k=1}^{m}$ is a continuous spline.

See also Ito and Teglas (1987) and Breda, Maset, and Vermiglio (2005).

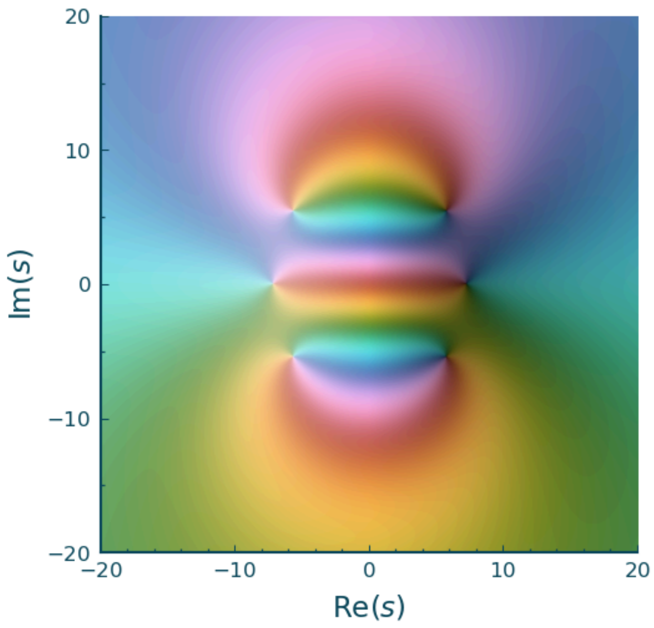
Convergence



$$\tau_1 = 1, \;\; \tau_2 = 1.9$$

In the frequency domain

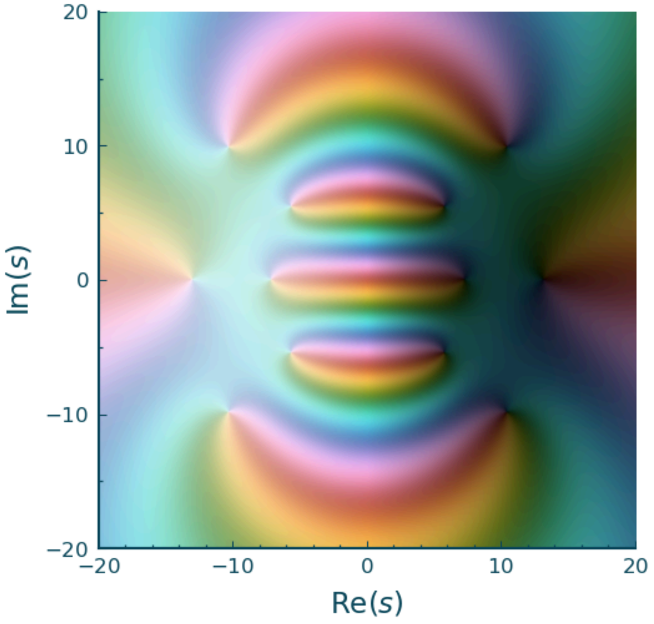
$$G_N^{\text{spl}}(s) = C\left(sI_n - \sum_{k=0}^{m} A_k r_N^{(k)}(s, -\tau_k)\right)^{-1} B$$



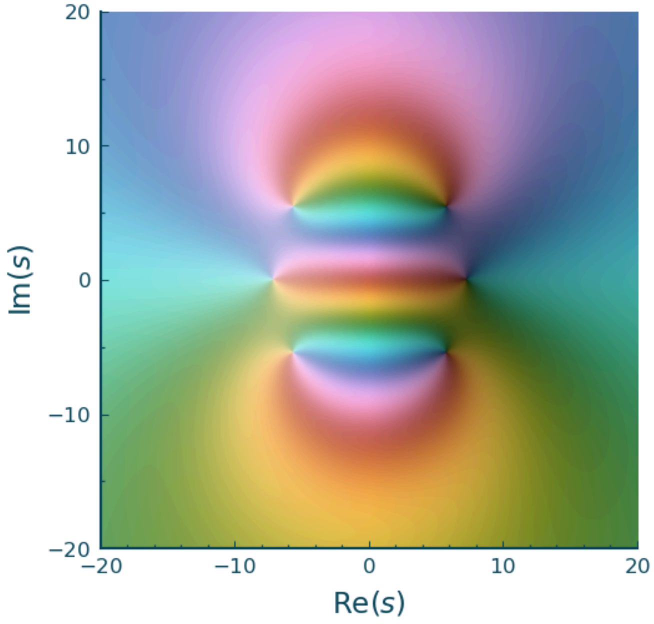
τ_1

In the frequency domain

$$G_N^{\text{spl}}(s) = C\left(sI_n - \sum_{k=0}^{m} A_k r_N^{(k)}(s, -\tau_k)\right)^{-1} B$$

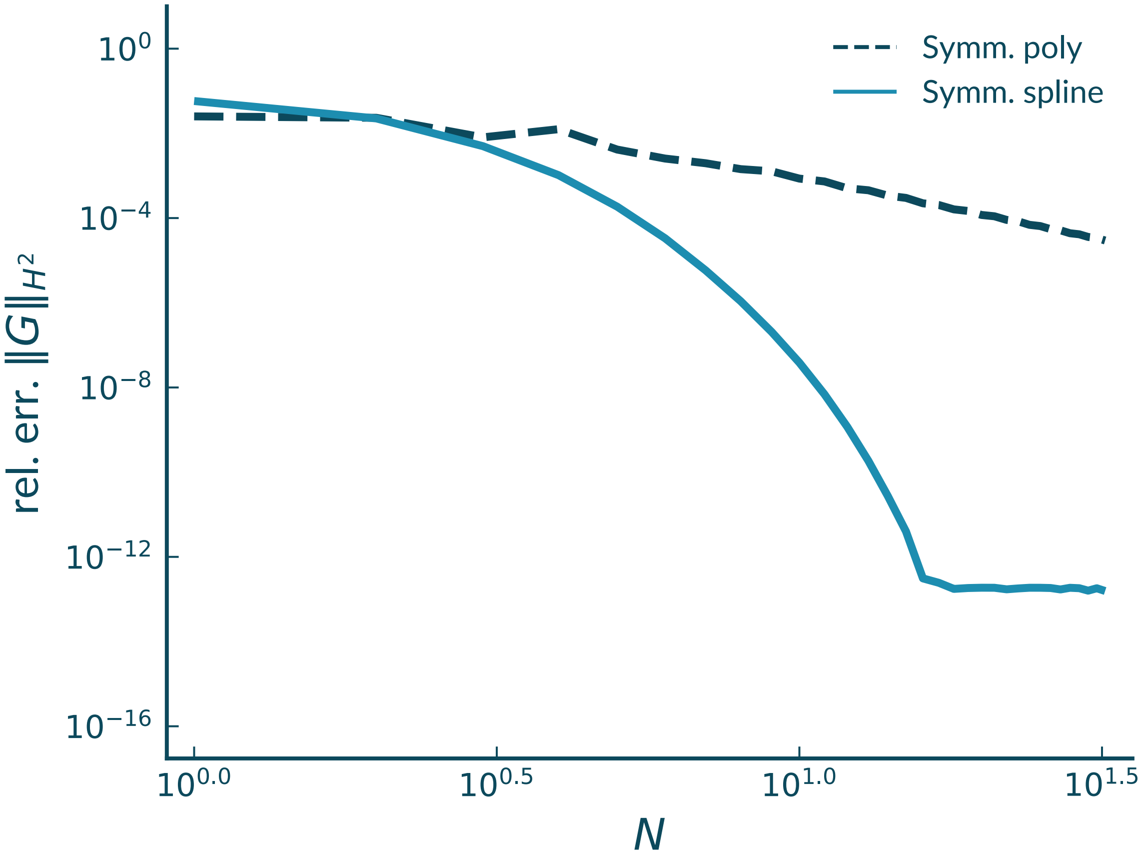


τ_2



τ_1

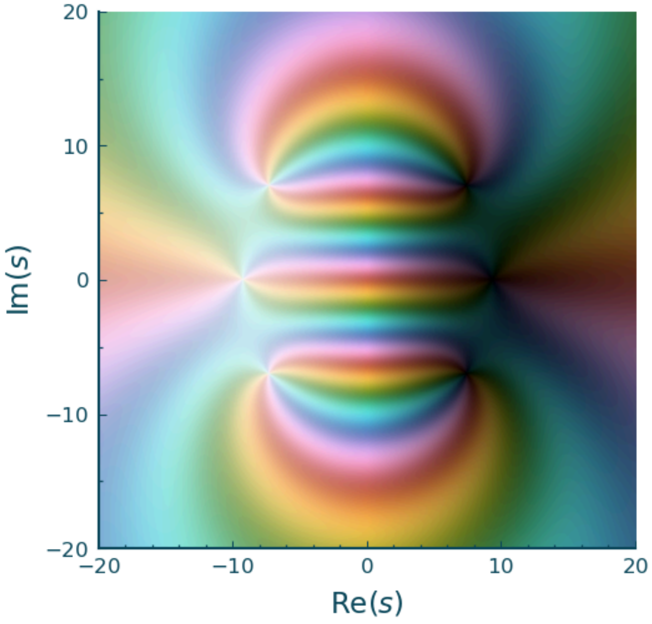
Convergence (equidistant)



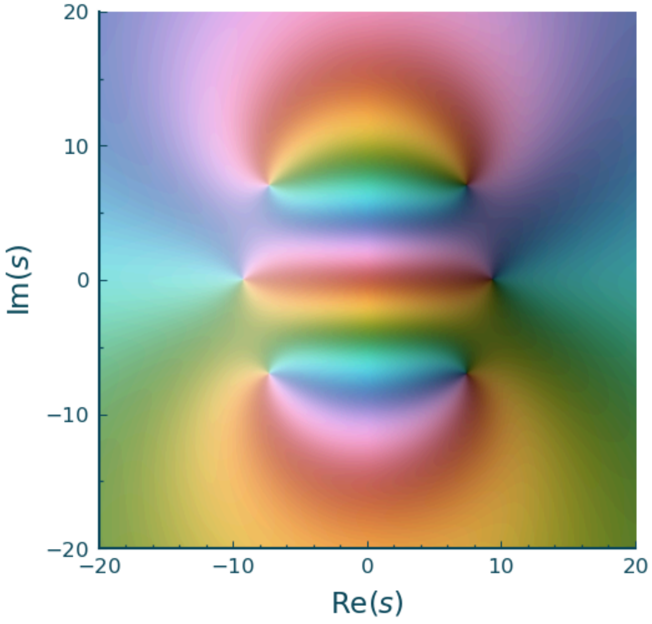
$$\tau_1 = 1, \ \ \tau_2 = 2$$

In the frequency domain

$$G_N^{\mathrm{spl}}(s) = C\Big(sI_n - \textstyle\sum_{k=0}^{m} A_k r_N^{(k)}(s, -\tau_k)\Big)^{-1} B$$



$\tau_2 = 2\tau_1$



τ_1

References

Breda, D., S. Maset, and R. Vermiglio. 2005. 'Pseudospectral Differencing Methods for Characteristic Roots of Delay Differential Equations'. *SIAM J Sci Comput* 27 (2): 482–95.

Ito, K., and R. Teglas. 1986. 'Legendre-Tau Approximations for Functional-Differential Equations'. *SIAM J Control Optim* 24 (4): 737–59.

Ito, K., and R. Teglas. 1987. 'Legendre-Tau Approximation for Functional Differential Equations Part II: The Linear Quadratic Optimal Control Problem'. *SIAM J Control Optim* 25 (6): 1379–1408.

Provoost, E., and W. Michiels. 2024. 'The Lanczos Tau Framework for Time-Delay Systems: Padé Approximation and Collocation Revisited'. *SIAM J Numer Anal* 62 (6): 2529–48.

Vanbiervliet, J., W. Michiels, and E. Jarlebring. 2011. 'Using Spectral Discretisation for the Optimal $\mathcal{H}_2$ Design of Time-Delay Systems'. *Int J Control* 84 (2): 228–41.

Contributions

Partially extended super-geometric convergence of the H^2-norm to multiple discrete delays using splines.

Attained super-geometric convergence for commensurate delays.

Showed that the underlying approximation of $e^{-\tau_k s}$ uses additional poles in each earlier delay intervals.

Derived explicit expressions for this rational approximation.*

Showed that $|r_N^{(k)}(i\omega, -\tau_k)| = 1$ for all k and $\omega \in \mathbb{R}$.*

*Not covered in this talk.